Scale relativity and non-differentiable fractal space-time

Laurent Nottale*
CNRS UMR 8631, DAEC, Observatoire de Paris-Meudon
F-92195 Meudon Cedex, France
E-mail: laurent.nottale@obspm.fr

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1 Introduction

The theory of scale relativity [14] is an attempt to study the consequences of giving up the hypothesis of space-time differentiability. One can show [14] [15] that a continuous but nondifferentiable space-time is necessarily *fractal*. Here the word fractal [12] is taken in a general meaning, as defining a set, object or space that shows structures at all scales, or on a wide range of scales. More precisely, one can demonstrate [17] that a continuous but nondifferentiable function is explicitly resolution-dependent, and that its length $\mathcal{L}$ tends to infinity when the resolution interval tends to zero, i.e. $\mathcal{L} = \mathcal{L}(\varepsilon)_{\varepsilon \to 0} \to \infty$. This theorem and other properties of non-differentiable curves have been recently analysed in detail by Ben Adda and Cresson [4]. It naturally leads to the proposal that the concept of *fractal space-time* [21] [25] [14] [7] is the geometric tool adapted to the research of such a new description. In such a generalized framework including all continuous functions, the usual differentiable functions remain included, but as very particular and rare cases.

Since a nondifferentiable, fractal space-time is explicitly resolution-dependent, the same is a priori true of all physical quantities that one can define in its framework. We thus need to complete the standard laws of physics (which are essentially laws of motion in classical physics) by laws of scale, intended to describe the new resolution dependence. We have suggested [13] that the principle of relativity can be extended to constrain also these new scale laws.

Namely, we generalize Einstein's formulation of the principle of relativity, by requiring *that the laws of nature be valid in any reference system, whatever its state.* Up to now, this principle has been applied to changes of state of the coordinate system that concerned the origin, the axes orientation, and the motion (measured in terms of velocity and acceleration). In scale relativity, we assume that the space-time resolutions are not only a characteristic of the measurement apparatus, but

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acquire a universal status. They are considered as essential variables, inherent to the physical description. We define them as characterizing the "state of scale" of the reference system, in the same way as the velocity characterizes its state of motion. The principle of scale relativity consists of applying the principle of relativity to such a scale-state. Then we set a principle of *scale-covariance*, requiring that the equations of physics keep their form under resolution transformations.

In the present paper, we shall review various levels of development of the theory, then consider some of its consequences in the domains of elementary particles, cosmology and gravitational structure formation.

2 Galilean scale relativity

2.1 Standard fractal laws

As we shall first see, simple fractal scale-invariant laws can be identified with a "Galilean" version of scale-relativistic laws. Indeed, let us consider a non-differentiable coordinate $\mathcal{L}$. Our basic theorem that links non-differentiability to fractality implies that $\mathcal{L}$ is an explicit function $\mathcal{L}(\varepsilon)$ of the resolution interval ε. As a first step, one can assume that $\mathcal{L}(\varepsilon)$ satisfies the simplest possible scale differential equation one may write, namely, the first order equation:

$$\frac{d \ln \mathcal{L}}{d \ln(\lambda/\varepsilon)} = \delta, \tag{1}$$

where δ is a constant. The solution is a fractal, power-law dependence:

$$\mathcal{L} = \mathcal{L}_0 (\lambda/\varepsilon)^{\delta}, \tag{2}$$

where δ is the scale dimension, i.e., $\delta = D - D_T$, the fractal dimension minus the topological dimension. The Galilean structure of the group of scale transformation that corresponds to this law can be verified in a straightforward manner from the fact that it transforms in a scale transformation $\varepsilon \to \varepsilon'$ as

$$\ln \frac{\mathcal{L}(\varepsilon')}{\mathcal{L}_0} = \ln \frac{\mathcal{L}(\varepsilon)}{\mathcal{L}_0} + \delta(\varepsilon) \ln \frac{\varepsilon}{\varepsilon'} \quad ; \quad \delta(\varepsilon') = \delta(\varepsilon). \tag{3}$$

This transformation has exactly the structure of the Galileo group, as confirmed by the law of composition of dilations $\varepsilon \to \varepsilon' \to \varepsilon''$, which writes $\ln \rho'' = \ln \rho + \ln \rho'$, with $\rho = \varepsilon'/\varepsilon$, $\rho' = \varepsilon''/\varepsilon'$ and $\rho'' = \varepsilon''/\varepsilon$.

2.2 Breaking of the scale symmetry

More generally, one can write a first order equation where the scale variation of $\mathcal{L}$ depends on $\mathcal{L}$ only, $d\mathcal{L}/d \ln \varepsilon = \beta(\mathcal{L})$. The function $\beta(\mathcal{L})$ is a priori unknown but, always taking the simplest case, we may consider a perturbative approach and take its Taylor expansion. We obtain the equation:

$$\frac{d\mathcal{L}}{d \ln \varepsilon} = a + b\mathcal{L} + \dots \tag{4}$$

This equation is solved in terms of a standard power law of power $\delta = -b$, broken at some relative scale λ (which is a constant of integration):

$$\mathcal{L} = \mathcal{L}_0 \, [1 + \left(\frac{\lambda}{\varepsilon}\right)^{\delta}]. \tag{5}$$

Depending on the sign of δ, this solution represents either a small-scale fractal behavior (in which the scale variable is a resolution), broken at larger scales, or a large-scale fractal behavior (in which the scale variable ε would now represent a changing window for a fixed resolution λ), broken at smaller scales.

2.3 Euler-Lagrange scale equations

In the previous approach, we have considered as primary variables the position $\mathcal{L}$ and the resolution ε. However, we are naturally led, in the scale-relativistic approach, to reverse the definition and the meaning of variables. The scale dimension δ can be generalized in terms of an essential, fundamental *variable*, that would remain constant only in very particular situations (namely, in the case of scale invariance, that corresponds to "scale-freedom"). It plays for scale laws the same role as played by time in motion laws. We have proposed to call "djinn" this varying scale dimension. The new approach amounts to work in a "space-time-djinn" rather than only in space-time, thus including motion and scale behaviour in the same 5-dimensional description. The resolution can now be defined as a derived quantity in terms of the fractal coordinate and of the djinn:

$$\bar{V} = \ln(\lambda/\varepsilon) = \frac{d \ln \mathcal{L}}{d\delta}. \tag{6}$$

Our identification of standard fractal behavior as Galilean scale laws can now be fully proven. We assume that, as in the case of motion laws, scale laws can be constructed from a Lagrangian approach. A scale Lagrange function $\bar{L}(\ln \mathcal{L}, \bar{V}, \delta)$ is introduced, from which a scale-action is constructed:

$$\bar{S} = \int_{\delta_1}^{\delta_2} \bar{L}(\ln \mathcal{L}, \bar{V}, \delta) d\delta. \tag{7}$$

The action principle, applied on this action, yields a scale-Euler-Lagrange equation

$$\frac{d}{d\delta}\frac{\partial \bar{L}}{\partial \bar{V}} = \frac{\partial \bar{L}}{\partial \ln \mathcal{L}}. \tag{8}$$

The simplest possible form for the Lagrange function is the equivalent for scales of what inertia is for motion, i.e., $\bar{L} \propto \bar{V}^2$ and $\partial \bar{L}/\partial \ln \mathcal{L} = 0$ (no scale "force"). The Lagrange equation writes in this case:

$$\frac{d\bar{V}}{d\delta} = 0 \Rightarrow \bar{V} = cst. \tag{9}$$

The constancy of $\bar{V} = \ln(\lambda/\varepsilon)$ means here that it is independent of the scale-time δ. Then Eq. (6) can be integrated in terms of the usual power law behavior, $\mathcal{L} = \mathcal{L}_0(\lambda/\varepsilon)^{\delta}$. This reversed viewpoint has several advantages which allow a full

implementation of the principle of scale relativity:
(i) The scale dimension takes its actual status of "scale-time", and the logarithm of resolution $\bar{V}$ its status of "scale-velocity", $\bar{V} = d\ln \mathcal{L}/d\delta$.
(ii) This leaves open the possibility of generalizing our formalism to the case of four independent space-time resolutions. Indeed, from $\mathcal{L}^{\mu}, \mu = 1, 2, 3, 4$ and δ one can now build a 4-component resolution vector, $\bar{V}^{\mu} = \ln(\lambda^{\mu}/\varepsilon^{\mu}) = d\ln \mathcal{L}^{\mu}/d\delta$.
(iii) At an even more profound level, one can jump to a 5-dimensional covariant representation (the "space-time-djinn") in which the djinn δ is given rank 0 and the four fractal space-time coordinates ranks 1 to 4. This leads to a 10 parameter rotation group in which we recover 3 rotations in space (xy, yz and zx), 3 Lorentz boosts (xt,yt and zt) and 4 resolution transformations (tδ,xδ,yδ,zδ). These new symetries lead to the emergence of four new conservative quantities (momenta), which we tentatively identify with the charges (see next section).

3 Special and generalized scale-relativity

3.1 Special scale relativity

It is well known that the Galileo group of motion is only a degeneration of the more general Lorentz group. The same is true for scale laws. Indeed, if one looks for the general linear laws of scale that come under the principle of scale relativity, one finds that, once they are expressed in logarithm form, they have the structure of the Lorentz group [13]. Therefore, in special scale relativity, we have suggested to substitute to the Galilean law of composition of dilations $\ln(\varepsilon'/\lambda) = \ln\rho + \ln(\varepsilon/\lambda)$ the more general log-Lorentzian law:

$$\ln\frac{\varepsilon'}{\lambda} = \frac{\ln\rho + \ln(\varepsilon/\lambda)}{1 + \ln\rho\ln(\varepsilon/\lambda)/\ln^2(\lambda_P/\lambda)}\,. \tag{10}$$

The scale dimension (or "djinn", δ) becomes itself a variable. More precisely, the couple it makes with the fractal coordinate $[\delta, \ln(L/\lambda)]$ becomes a scale vector. In the simplified case when $\delta(\lambda) = 1$ (i.e., fractal dimension 2 at transition scale) and $L(\lambda) = \lambda$, it writes:

$$\delta(\varepsilon) = \frac{1}{\sqrt{1 - \ln^2(\varepsilon/\lambda)/\ln^2(\lambda_P/\lambda)}}, \tag{11}$$

where λ is the fractal-nonfractal transition scale (e.g., the Compton length of a particle). In such a law, there appears a minimal (or maximal) scale of space-time resolution which is invariant under dilations and contractions, and plays the same role for scales as that played by the velocity of light for motion.

Toward the small scales, this invariant length-scale is naturally identified with the Planck scale, $\lambda_P = (\hbar G/c^3)^{1/2}$, that now becomes impassable and plays the physical role that was previously devoted to the zero point. The same is true in the cosmological domain, with an inversion of the scale laws: there appears a maximal, impassable scale of resolution that plays the physical role of the infinite, that we have identified with the length-scale $\mathbb{L} = \Lambda^{-1/2}$, where Λ is the cosmological constant .

Some consequences of this new interpretation of the Planck length-time-scale have been considered elsewhere [13][15], concerning in particular the unification of fundamental fields. Let us briefly discuss here its consequence as concerns the status of units. We already know that special motion-relativity has changed the status of space and time units. Indeed, the very existence of space-time implies to use the same units for length and time intervals. This is achieved since 1985, the unit of length being derived from the unit of time, and c fixed. Therefore, the genuine nature of velocities is adimensional pure numbers always smaller than one. A new step can be made in this direction using special scale relativity. Indeed, in its framework, every length (or time) interval is written in terms of its ratio over the Planck length (time)-scale. The Planck length and time scales thus appear as natural units of length and time intervals, whose genuine nature is found to be pure, adimensional numbers larger than one. In the end, this implies that space and time units do not really exist, since, in the same way as the limitation of 3-velocity is a pure effect of projection from 4-space-time to 3-space, the Planck limit is the simple result of projection from 5 dimensional to 4 dimensional space. More generally, if one replaces the three fundamental constants G, $\hbar$ and c by their expressions in terms of the Planck time, length and mass in any equation of physics, all quantities appear in these equations in terms of their ratio over the corresponding Planck units, which, ultimately, vanish from physics. As a consequence, the time is certainly come to re-found our system of units, in particular concerning masses, which should be referred to the Planck mass ($\approx 2 \times 10^{-5}$ g).

3.2 Scale-motion coupling and mass-charge relations

The theory of scale relativity also allows to get new insights about the physical meaning of gauge invariance [15]. In the scale laws recalled hereabove, only scale transformations at a given point were considered. But we may also wonder about what happens to the structures in scale-space of a scale-dependent object such as an electron or another charged particle, when it is displaced. Consider anyone of these structures, lying at some (relative) resolution ε (such that $\varepsilon < \lambda$, where λ is the Compton length of the particle) for a given position of the particle. In a displacement, the relativity of scales implies that the resolution at which this given structure appears in the new position will a priori be different from the initial one. In other words, $\varepsilon = \varepsilon(x,t)$ is now a function of the space-time coordinates, and we expect the occurrence of *dilations of resolutions induced by translations*, so that we are led to introduce a covariant derivative:

$$e\,\frac{D\varepsilon}{\varepsilon} = e\,\frac{d\varepsilon}{\varepsilon} - A_\mu dx^\mu, \tag{12}$$

where a four-vector A_μ must be introduced since dx^μ is itself a four-vector and $d \ln \varepsilon$ a scalar (in the case of a global dilation).

However, if one wants such a "field" A_μ to be physical, it should be defined whatever the initial scale from which we started. Starting from another scale $\varepsilon' = \rho\varepsilon$, we get the same expression as in Eq.(12), but with A_μ replaced by A'_μ. Therefore, we obtain the relation:

$$A'_\mu = A_\mu + e\,\partial_\mu \ln \rho, \tag{13}$$

which depends on the relative "state of scale", $\bar{V} = \ln \rho = \ln(\varepsilon/\varepsilon')$, that is now a function of the coordinates.

One may therefore identify A_μ with the electromagnetic potential, and Eq.(13) with the property of gauge invariance. Now we know that applying a gauge transformation to the electromagnetic field implies to change also the wave function of the electron, that becomes:

$$\psi' = \psi \, e^{i 4\pi\alpha \ln\rho} \tag{14}$$

where α is a coupling constant. While in Galilean scale relativity, the scale ratio ρ is unlimited, in the more general framework of special scale relativity it is limited by the Planck-scale/Compton-scale ratio. This limitation implies the quantization of charge, following a general mass-charge relation [15]: $\alpha \ln(m_P/m) = k/2$, where k is integer. In order to compare such a relation with experimental data, one should account for the electroweak theory, according to which the electromagnetic coupling is only 3/8 of its high energy value (plus radiative corrections). We get:

$$\frac{8}{3}\,\alpha_{em}\,\ln\left(\frac{m_P}{m_e}\right) = 1 \tag{15}$$

where $\alpha_{em} = 1/137.036$ is the low energy fine structure constant and m_e is the electron mass. This relation is implemented with a relative precision of 2×10^{-3}, becoming 10^{-4} when accounting for threshhold effects [15].

This approach can be generalized, since, as recalled hereabove, we can define four different and independent dilations along the four space-time resolutions instead of only one global dilation. The above U(1) field is then expected to be embedded into a larger field, in agreement with the electroweak and grand unification theories, and the charge e to be one element of a more complicated, "vectorial" charge. Some hints about such a generalization will be given in what follows.

More generally, we shall be led to look for the general non-linear scale laws that satisfy the principle of scale relativity. Such a generalized framework implies working in a five-dimensional space-time-djinn. The development of such a "general scale-relativity" lies outside the scope of the present paper and will be considered in forthcoming works.

4 Theoretical predictions of masses and couplings

In the new framework, theoretical predictions of some of the free parameters of the standard model become possible. We have presented and checked such predictions in previous works [13] [14] [15]. But in the recent years, there has been an improvement of several experimental measurements [28], so that it may now be interesting to check them again with these new values. They are, respectively for the top quark mass, Higgs boson mass, W and Z boson masses, strong coupling constant at Z scale, fine structure constant at Z scale, and $\sin^2\theta$ of weak mixing angle at Z scale in the modified minimal substraction scheme (where it is defined through the SU(2) charge g and the U(1) charge g'):

$m_t = 174.3 \pm 5.1\,\mathrm{GeV}$; $m_H = 108 - 220\,\mathrm{GeV}$
$m_W = 80.42 \pm 0.04\,\mathrm{GeV}$; $m_Z = 91.1872 \pm 0.0021$
$\alpha_S(m_Z)^{-1} = 0.118 \pm 0.002$; $\alpha(m_Z)^{-1} = 128.92 \pm 0.03$
$\hat{s}_Z^2 = \frac{g'^2}{g^2+g'^2} = 0.23117 \pm 0.00016$

4.1 Fine structure constant

In [15], we derived a prediction of the fine structure constant (i.e. the electromagnetic coupling). It was based on the suggestion that the bare (infinite energy) value of the electroweak coupling (which becomes finite in special scale-relativity) is $4\pi^2$. The fact that 3 among the 4 gauge bosons acquire mass through the Higgs mechanisms leads to a multiplying factor 8/3, so that one expects that $\alpha_\infty^{-1} = 32\pi^2/3$. The difference between the infinite energy and Z or low energy values was computed using the solutions to the renormalization group equation for the running coupling. The prediction at the Z value for 1 Higgs doublet is (see more detail in [15] [20]):

$$\alpha(m_Z)^{-1} = \frac{32\pi^2}{3} + \frac{22\pi}{3} + \frac{6}{\pi^2} = 128.922\,. \tag{16}$$

This result compares very well with the experimental value, 128.92 ± 0.03.

4.2 Strong coupling

From the conjecture that the strong coupling value reaches the critical value $1/4\pi^2$ at unification scale (i.e. $m_P/2\pi$ in the special scale-relativistic modified standard model), we obtained a predicted value $\alpha_S(m_Z)^{-1} = 0.1155 \pm 0.0002$ from the solution to the renormalization group equation of the running coupling [13] [15]. This expectation remains in agreement (within about one σ) with the recently improved experimental value 0.118 ± 0.002.

4.3 SU(2) coupling

In [15], we also attempted to apply the mass-charge relation to the SU(2) coupling α_2. We found that the relation

$$3\,\alpha_{2Z}\,C_Z = 4 \tag{17}$$

was precisely achieved at the Z scale. However the factor 3 was not accounted for in that work. The solution to this problem relies on the generalization of scale (i.e. gauge) transformations to dilations which are no longer global, but instead may be different on the internal resolutions corresponding to the various coordinates. The group SU(2) corresponds to rotations in a 3-dimensional scale space. Therefore the phase term in a fermion field will write:

$$\alpha_2 \ln(\frac{\varepsilon_x}{\lambda}) + \alpha_2 \ln(\frac{\varepsilon_y}{\lambda}) + \alpha_2 \ln(\frac{\varepsilon_z}{\lambda}) < 3\,\alpha_2 \ln(\frac{\lambda_P}{\lambda}), \tag{18}$$

since the same coupling applies to the three variables, and since all three resolutions are limited at small scales by the Planck scale. From Eq. (17) we expect a value $\alpha_{2Z}^{-1} = 29.8169 \pm 0.0002$. The present precise experimental value is:

$$\alpha_{2Z}^{-1} = \alpha_Z^{-1} \times \hat{s}_Z^2 = 29.802 \pm 0.027, \tag{19}$$

which lies within 1σ of the theoretical prediction.

4.4 Vacuum expectation value of the Higgs field

As recalled hereabove, there are fundamental arguments for introducing a bare inverse coupling at infinite energy (i.e., in special scale relativity, at Planck length-scale) given by the critical value $4\pi^2$. Moreover, our re-interpretation of gauge invariance as scale-invariance on space-time resolution led us to construct general relations between couplings and scale ratios. Therefore one expects the emergence of a new fundamental scale given by:

$$\ln\left(\frac{\lambda}{\lambda_P}\right) = 4\pi^2, \tag{20}$$

where λ_P is the Planck length-scale. This relation may provide a solution to the hierarchy problem, according to which there is a misunderstood factor $\approx 10^{17}$ between the electroweak scale and the Planck scale (expected to be the full unification scale). Indeed the length-scale λ defined above is $e^{4\pi^2} = 1.397 \times 10^{17}$ larger than the Planck scale. As a first approximation, we can apply this relation to mass ratios. This gives a mass scale of 87.39 GeV, intermediate between the Z and W masses. However, mass-scales and length-scales are no longer directly inverse in the scale-relativity framework. There is a "log-Lorentz" factor between them (when they are referred to low energies). Namely, by taking as reference the electron Compton scale, the new mass-scale is more precisely given by:

$$\ln\left(\frac{m}{m_e}\right) = \frac{\ln(\lambda_e/\lambda)}{\sqrt{1-\ln^2(\lambda_e/\lambda)/C_e^2}}. \tag{21}$$

With the currently accepted value of the gravitational constant (for which the error is now thought to be 12 times larger than previously given, see [28]), we obtain for the fundamental constant $C_e = \ln(\lambda_e/\lambda_P) = 51.52797(70)$. Then the new theoretically predicted mass scale is

$$m_v = 123.23 \pm 0.09\,\mathrm{GeV}, \tag{22}$$

which is closely linked to the vacuum expectation value v of the Higgs field, since the present experimental value of $v/\sqrt{2} = m_W/g$ (where g is the SU(2) weak charge) is $123.11 \pm 0.03\,\mathrm{GeV}$. Now some work remains to be done to really understand why the new mass-scale should have precisely this interpretation.

4.5 Mass of the Higgs boson

The framework of generalized scale-relativity provides one with possibilities to make theoretical predictions of the value of the Higgs boson mass. The (summarized) argument is as follows.

In today's electroweak scheme, the Higgs boson is considered to be separated from the electroweak field. Moreover, a more complete unification is mainly seeked in terms of attempts of "grand" unifications with the strong field. However, one may wonder whether, maybe in terms of an effective theory at intermediate energy, one could achieve a more tightly unified purely electroweak theory. Recall indeed that in the present standard model, the weak and electromagnetic fields are mixed, but there remains four free parameters, which can e.g. be taken to be the Higgs boson mass, the vacuum expectation value of the Higgs field and the Z and W masses. In the attempt sketched out hereafter, the Higgs field is assumed to be a part of the total field, so that only two free parameters would be left. As a consequence, the Higgs boson mass and the W/Z mass ratio could be derived in such a model.

Recall that the structure of the present electroweak boson content is as follows. There is a SU(2) gauge field, then involving three fields of null mass (i.e. $2 \times 3 = 6$ degrees of freedom), a U(1) null mass field (2 d.f.) and a Higgs boson complex doublet (4 d.f.), which makes 12 degrees of freedom in all. Through the Glashow-Salam-Weinberg mechanism, 3 of the 4 components of the Higgs doublet become longitudinal components of the weak field which therefore acquires mass ($3 \times 3 = 9$ d.f.), while the photon remains massless (2 d.f.), so that there remains a Higgs scalar which is nowadays experimentally searched (1 d.f.).

Now, we have suggested a new interpretation of gauge invariance as being scale invariance on the internal resolutions, considered as intrinsic to the description of the particle-fields (at scale smaller than their Compton length in restframe). As a first step we considered only global dilations, which led us to a U(1) invariance and to the relations between mass scale and coupling constant recalled above. But more generally one may consider four independant scale transformations on the four space-time resolutions, i.e., $(\ln \varepsilon_x, \ln \varepsilon_y, \ln \varepsilon_z, \ln \varepsilon_t)$. This means that the scale space (i.e., here the gauge space) is at least four-dimensional (but note that this is not the final word on the subject, since this does not yet include the fifth "djinn" dimension δ). Moreover, the mixing relation between the B [U(1)] and W_3 [SU(2)] fields may also be interpreted as a rotation in the full gauge space. Therefore we expect the appearance of a 6 component antisymmetric tensor field (linked to the rotations in this space), corresponding in the simplest case to a SO(4) group. Such a zero mass field would yield 12 degrees of freedom by itself alone, so that it is able to include the electromagnetic and weak fields, but also the residual Higgs field.

What about the Higgs boson in such a unified framework ? We shall tentatively explore the possibility that it appears as a separated scalar only as a low energy approximation, while in the new framework it would be one of the components of the unified field (in analogy with energy appearing as scalar at low velocity, while it is ultimately a component of the energy-momentum four-vector).

Such an attempt is supported by the form of the electroweak Lagrangian (we adopt Aitchison's [3] notations). Its Higgs scalar boson part writes:

$$L_H = \frac{1}{2}\, \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2}\, m_H^2 \sigma^2 - \frac{1}{8} \lambda^2 \sigma^4. \tag{23}$$

The vacuum expectation value v of the Higgs field is computed from the square (mass term) and quartic term, so that the Higgs mass is related to v and λ as:

$$m_H = \sqrt{2}\, v\, \lambda. \tag{24}$$

A prediction of the constant λ would therefore lead to a prediction of the Higgs mass. Now, a non-Abelian field writes in terms of its potential :

$$F^{\alpha\mu\nu} = \partial^\mu W^{\alpha\nu} - \partial^\nu W^{\alpha\mu} - g\, c^\alpha_{\beta\gamma} W^{\beta\mu} W^{\gamma\nu}, \tag{25}$$

where g is the (now unique) charge and $c^{\alpha}_{\beta\gamma}$ the structure coefficients of the Lie algebra associated to the gauge group. Its Lagrangian writes:

$$L_W = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}. \tag{26}$$

Therefore, it includes W^4 terms coming from the W^2 terms in the field. Now our ansatz consists of identifying some of these W^4 terms, of coefficient $-\frac{1}{4}\, g^2\, (c^{\alpha}_{\beta\gamma})^2$, with the Higgs boson σ^4 term of coefficient $-\frac{1}{2}\, \lambda^2$.

Namely, let us first separate the six components of the total field in two subsystems, $[W_1, W_2, W_3]$ and $[B_1, B_2, B_3]$. The three W's can be identified with the standard SU(2) field and, say, B_1 with the $\mathrm{U(1)_Y}$ field.Two vectorial fields remain, B_2^{μ} and B_3^{μ}. They will contribute in a non-vanishing way to the quartic term in the Lagrangian by their cross product. At the approximation (considered here) where their space components are negligible, we find:

$$-B_{2\mu}\, B_{3\nu}\, B_3^{\mu}\, B_2^{\nu} = -[B_2^0\, B_3^0]^2. \tag{27}$$

Finally, we make the identification of these time components with the residual Higgs boson, $B_2^0 = B_3^0 = \sigma$. This allows a determination of the constant λ according to the relation:

$$\lambda^2 = \frac{g^2\, c^2}{2}, \tag{28}$$

where the squared Lie coefficient $c^2 = 1$ in the case of an SO(4) group. Provided the global charge is identical to the SU(2) charge, and since the W mass is given by $m_W = gv/\sqrt{2}$, one finally obtains a Higgs boson mass :

$$m_H = \sqrt{2c^2}\, m_W. \tag{29}$$

More generally, one must make the sum of all the terms that contribute to the final Higgs boson, and since the c's take the values 0, ± 1 for a large class of groups, we expect $m_H = \sqrt{2k}\, m_W$ with k integer. In particular, the simplest case $k = 1$ yields a theoretical prediction [20]:

$$m_H = \sqrt{2}\, m_W = 113.73 \pm 0.06\, \mathrm{GeV}, \tag{30}$$

which is in agreement with current constraints and with a possible recent detection at CERN. Although this calculation is still incomplete and although the self-consistency of this model remains to be established, we hope that at least some of its ingredients could reveal to be useful in more complete attempts [Lehner and Nottale, in preparation].

4.6 Cosmological constant and vacuum energy density

In [15], we were able to make a theoretical prediction of the value of the cosmological constant. Recall that, in the special scale-relativistic framework, new dilation laws having a log-Lorentz form have been introduced [13], that lead to re-interpret the length-scale $\mathbb{L} = \Lambda^{-1/2}$, where Λ is the cosmological constant, as an impassable, maximal resolution scale, and the Planck length-scale λ_P as a minimal length-scale, invariant under dilations of resolutions.

One of the most difficult open questions in present cosmology is the problem of the vacuum energy density and of its manifestation as an effective cosmological constant [33][6]. The scale relativity approach generalizes Zeldovich's [34] approach. It allows one to suggest a solution to this problem and to connect it with Dirac's large number hypothesis (see also Sidharth [32]). The first step toward our solution consists in considering the vacuum as fractal, (i.e., explicitly scale dependent). As a consequence, the Planck value of the vacuum energy density (that gave rise to the 10^{120} discrepancy with observational limits) is relevant only at the Planck scale, and becomes irrelevant at the cosmological scale. We expect the vacuum energy density ρ to be solution of a (renormalisation group-like) scale differential equation:

$$\frac{d\rho}{d\ln r} = \Gamma(\rho) = a + b\rho + O(\rho^2), \tag{31}$$

where ρ has been normalized to its Planck value, so that it is always < 1, allowing us to perform a Taylor expansion of $\Gamma(\rho)$. This equation is solved as:

$$\rho = \rho_c\,[1 + \left(\frac{r_0}{r}\right)^{-b}], \tag{32}$$

where $\rho_c = -a/b$ can be identified with the cosmological energy density. We recover the well-known combination of a power law behavior at small scales and of scale-independence at large scale, with a fractal/non-fractal transition about some scale r_0 that comes out as an integration constant.

The second step toward a solution is to realize that, when considering the various field contributions to the vacuum density, we may always chose $< E >= 0$ (i.e., renormalize the energy density of the vacuum). But consider now the gravitational self-energy of vacuum fluctuations. It writes:

$$E_g = \frac{G}{c^4}\,\frac{< E^2 >}{r}. \tag{33}$$

The Heisenberg relation prevent from making $< E^2 >= 0$, so that this gravitational self-energy cannot vanish. This relation writes $< E^2 >^{1/2}= \hbar c/r$, so that we obtain the asymptotic high energy behavior:

$$\rho_g = \rho_P \left(\frac{\lambda_P}{r}\right)^6, \tag{34}$$

where λ_P is the Planck length and ρ_P is the Planck energy density. From this equation we can make the identification $b = -6$. Therefore we obtain the complete behavior of the vacuum energy density:

$$\rho = \rho_c\,[1 + \left(\frac{r_0}{r}\right)^6]. \tag{35}$$

For $r >> r_0$, it becomes invariant and is equal to the cosmological energy density ρ_c which manifest itself as a cosmological constant, while for $r << r_0$ it varies very rapidly with scale. We have recovered here, in another way, the statement according to which the vacuum energy density may be the sum of a constant, geometrical, large scale term and of a quantum term (here considered as scale dependent). Moreover, the value of the large scale energy density is given (up to a factor of 2) by the value it reaches at the transition scale r_0, so that the problem of determining the cosmological constant now amounts to determining this length-scale.

We are now able to demonstrate one of Eddington-Dirac's large number relations, and to write it in terms of invariant quantities (i.e., we do not need varying constants to implement it in this form).

Indeed, introducing the maximal scale-relativistic length scale $\mathbb{L} = \Lambda^{-1/2}$, we get the relation:

$$\mathbb{K} = \frac{\mathbb{L}}{\lambda_P} = \left(\frac{r_0}{\lambda_P}\right)^3 = \left(\frac{m_P}{m_0}\right)^3, \tag{36}$$

where r_0 is the Compton length associated with the mass scale m_0. Then the power 3 in Dirac's relation is understood as coming from the power 6 of the gravitational self-energy of vacuum fluctuations and of the power 2 that connects the invariant impassable scale $\mathbb{L}$ to the cosmological constant, following the relation $\Lambda = 1/\mathbb{L}^2$.

Now a complete solution to the problem would be reached only provided the transition scale r_0 be identified. We have suggested [15] that this scale be nothing but the QCD scale, i.e., that the final value of the cosmological constant is fixed at the quark-hadron transition during the Big-Bang. Indeed, before the epoch of this quark-hadron transition, the expansion of the Universe is concerned with free quarks. When the quark distance reaches the size of hadrons (about one Fermi), confinement begins to occur, quarks can no longer be subjected to the expansion, which now applies between hadrons. But, considering the vacuum, one may make the conjecture that the vacuum energy density outside hadrons has been "frozen" at the value it keeps inside hadrons.

Now the QCD scale for 6 quark flavours is found to be $\lambda_{QCD} = 66 \pm 10$ MeV. Note that several fundamental scales of physics fall very close to this energy: the classical radius of the electron, that yields the e^+e annihilation cross section at the energy of the electron mass and corresponds to an energy 70.02 MeV; the effective mass of quarks in the lightest meson, $m_\pi/2 = 69.78$ MeV; the diameter of nucleons, that corresponds to an energy 2×64 MeV. We have suggested to identify the transition scale r_0 with this particular scale and we have obtained [14] [15]:

$$\mathbb{K} = (5.3 \pm 2.0) \times 10^{60}, \tag{37}$$

allowing us to predict a cosmological constant $\Lambda = 1.36 \times 10^{56} cm^2$, i.e. a reduced cosmological constant

$$\Omega_\Lambda = 0.36\, h^{-2}, \tag{38}$$

where $h = H_0/100$ km/s.Mpc. Now the Hubble constant has been recently determined with an improved precision to be $H_0 = 70 \pm 10$ km/s.Mpc. Therefore our theoretical prediction yields a reduced cosmological constant $\Omega_\Lambda = 0.70 \pm 0.25$. Recent measurements using the Hubble diagram of SNe I [9] [29] [30] and the angular power spectrum of the cosmic microwave radiation [5] point precisely toward the same value, 0.7 ± 0.2.

5 Generalized Schrodinger equation

One can demonstrate [14] [15] [17] that, when giving up the hypothesis of differentiability of space-time coordinates, Newton's fundamental equation of dynamics can be integrated in the form of a Schrödinger-like equation. Indeed, as recalled in the introduction, a non-differentiable continuum is necessarily fractal, and trajectories in such a space (or space-time) own (at least) the following three properties:

(i) The test-particles can follow an infinity of potential trajectories: this leads one to use a fluid-like description, $v = v(x(t), t)$.

(ii) The geometry of each trajectory is fractal (of dimension 2). Each elementary displacement is then described in terms of the sum, $dX = dx + d\xi$, of a mean, classical displacement $dx = v\,dt$ and of a fractal fluctuation $d\xi$ whose behavior satisfies the principle of scale relativity (in its simplest "Galilean" version). It is such that $< d\xi >= 0$ and $< d\xi^2 >= 2\mathcal{D} dt$. The existence of this fluctuation implies introducing new second order terms in the differential equations of motion.

(iii) The motion is locally irreversible, i.e., the $(dt \leftrightarrow -dt)$ reflection invariance is broken, leading to a two-valuedness of the velocity vector that we represent in terms of a complex velocity, $\mathcal{V} = (v_+ + v_-)/2 - i(v_+ - v_-)/2$.

These three effects can be combined to construct a complex time-derivative operator which writes

$$\frac{\acute{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{V} \cdot \nabla - i\mathcal{D}\,\triangle \tag{39}$$

where the mean velocity $\mathcal{V} = \acute{d}x/dt$ is now complex and $\mathcal{D}$ is a parameter characterizing the fractal behavior of trajectories (namely, it defines the fractal-nonfractal transition in scale space).

Since the mean velocity is complex, the same is true of the Lagrange function, then of the generalized action $\mathcal{S}$. Setting $\psi = e^{i\mathcal{S}/2m\mathcal{D}}$, Newton's equation of dynamics becomes $m\,\acute{d}\mathcal{V}/dt = -\nabla\phi$, and can be integrated in terms of a generalized Schrödinger equation [14]:

$$\mathcal{D}^2 \triangle \psi + i\mathcal{D}\frac{\partial}{\partial t}\psi = \frac{\phi}{2m}\,\psi. \tag{40}$$

Since the imaginary part of this equation is the equation of continuity, $\rho = \psi\psi^\dagger$ can be interpreted as giving the probability density of the particle position.

Such an approach can be applied to standard quantum mechanics in the microphysical domain, but also, as an approximation, to macroscopic problems of gravitational structuration [17]. In this case, even though it takes this Schrödinger-like form, this equation is still an equation of gravitation, so that it must keep the fundamental properties it owns in Newton's and Einstein's theories. Namely, it must agree with the equivalence principle [16] [10] [1], i.e., it must be independent of the mass of the test-particle and GM must provide the natural length-unit of the system under consideration. As a consequence, the parameter $\mathcal{D}$ takes the form $\mathcal{D} = \frac{GM}{2w}$, where w is a fundamental constant that has the dimension of a velocity. Actually, the ratio $\alpha_g = \frac{w}{c}$ stands out as a macroscopic gravitational coupling constant [1] [2] [24].

It has been shown that this approach accounts for several structures observed in the Solar System [14], including planet distances, eccentricities, and mass distribution [23], obliquities and inclinations of planets and satellites [18]), giant planet satellite distances [11], parabolic comet perihelions [Nottale & Schumacher, in preparation]. Moreover, it also allows one to predict and understand structures observed on a large range of scales, from binary stars [22], to binary galaxies [15]

[Tricottet & Nottale, in preparation], and the distribution of galaxies at the scale of the local supercluster [22]. It has been also demonstrated that the first newly discovered extra-solar planetary systems come under the same structures, in terms of the same universal constant as in our own Solar System [16] [24] [19].

Let us finally remark that this theory could also help solving the problem of dark "matter". Recall that, by dynamical and gravitational lensing arguments, one observes the effects of a potential energy additional to that coming from visible matter, from the scale of galaxies to cosmological scales. In current attempts, this potential is tentatively attributed to missing matter. However, one can now suggest a different, more direct explanation. Indeed, by separating the real and imaginary parts of the generalized Schrödinger equation we get respectively a generalized Euler-Newton equation and the continuity equation (which is therefore now part of the dynamics):

$$m\left(\frac{\partial}{\partial t} + V \cdot \nabla\right)V = -\nabla(\phi + Q), \tag{41}$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho V) = 0. \tag{42}$$

Then this system of equations is equivalent to the classical one, except for the introduction of an extra potential energy term Q that writes:

$$Q = -2m\mathcal{D}^2 \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}. \tag{43}$$

This additional potential energy, (the so-called Böhm potential), can be understood in our framework as a manifestation of the fractal and non-differentiable geometry of space itself. We suggest that it may explain the various effects that have been attributed up to now to dark matter, since it contains a non-zero vacuum energy term. For example, for the fundamental level in a Kepler potential (the expected potential exterior to a galaxy), one finds a constant term $-\frac{1}{2}mw_0^2$ indicating a constant rotation velocity w_0 as observed. First trials of comparison with the observed behavior of these effects have given encouraging results [L.N., in preparation].

6 Conclusion

After having summarized the main lines of development of the theory, we have, in the present contribution, updated some of its theoretical predictions, then we have shown that they continue to agree with recently improved experimental results.

Moreover, we have recalled that scale relativity, when combined with the laws of gravitation, provides us with a general theory of the structuring of gravitational systems [15] [17]. This new approach is complementary of the standard one. In situations where we can no longer follow individual trajectories,we may jump to a statistical description in terms of probability amplitudes which are solutions of a generalized Schrödinger equation. This result suggests, in accordance with recent similar conclusions [31] [26] [27] [8] that the Schrödinger equation could be universal, i.e. that it may have a larger domain of application than previously thought, but with an interpretation different from that of standard quantum mechanics.

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