Scale Relativity and Fractal Space-Time: Applications to Quantum Physics, Cosmology and Chaotic Systems *

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Abstract

The theory of scale relativity is a new approach to the problem of the origin of fundamental scales and of scaling laws in physics, that consists of generalizing Einstein's principle of relativity (up to now applied to motion laws) to scale transformations. Namely, we redefine space-time resolutions as characterizing the state of scale of the reference system, and require that the equations of physics keep their form under resolution transformations (i.e. be scale-covariant). We recall in the present review paper how the development of the theory is intrinsically linked to the concept of fractal space-time, and how it allows one to recover quantum mechanics as mechanics on such a non-differentiable space-time, in which the Schrödinger equation is demonstrated as a geodesics equation. We recall that the standard quantum behavior is obtained, however, as a manifestation of a "Galilean" version of the theory, while the application of the principle of relativity to linear scale laws leads to the construction of a theory of special scale-relativity, in which there appears impassable, minimal and maximal scales, invariant under dilations. The theory is then applied to its preferential domains of applications, namely very small and very large length- and time-scales, i.e., high energy physics, cosmology and chaotic systems. Copyright ©1996 Elsevier Science Ltd

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Content

1 INTRODUCTION

The theory of scale relativity [1-3] is an attempt at tackling the various problems of scales encountered in today's physics by the construction of a new geometric framework in which motion laws are completed by scale laws. Its starting point is the realization that the whole of present physics relies on the unjustified assumption of differentiability of the space-time continuum. Even though this hypothesis seems to be well verified in the classical domain, it is clearly broken by the quantum mechanical behavior. Indeed it was demonstrated by Feynman [4] that the typical paths of quantum mechanical particles are continuous but non-differentiable.

Giving up the hypothesis of differentiability has an important physical consequence: one can show that curves, surfaces, volumes, more generally spaces of topological dimension D_T, which are continuous but nondifferentiable, are characterized by a length, an area, more generally a D_T -measure which becomes explicitly dependent on the resolution at which they are considered, $\mathcal{M} = \mathcal{M}(\varepsilon)$, and tends to infinity when the resolution interval ε tends to zero. In other words, a nondifferentiable space-time continuum is necessarily *fractal*, in the general meaning initially given to this word by Mandelbrot [5]. This result naturally leads to the proposal of a geometric tool adapted to construct a theory based on such premises, namely, fractal space-time [1,6,7,8].

The development of the theory first consists in making the various physical quantities explicitly dependent on the space-time resolutions. As a consequence the fundamental equations of physics will themselves become scale-dependent. This new dependence on scale will then be constrained by setting the postulate that these equations must be covariant under scale transformations of resolutions (i.e., their dilations and contractions).

Such a frame of thought extends Einstein's principle of relativity to scale transformations, while up to now it has been considered as applying only to motion transformations (more generally, to displacements in four-space-time). These new scale transformations of resolutions can be described, as we shall see, by rotations in a new five-dimensional 'space-time-zoom'. Scale covariance, as motion covariance, means that the equations of physics must keep their simplest form, i.e. after an arbitrarily complicated scale (or motion) transformation they must maintain the form they had in the simplest system of coordinate (see, e.g., Weinberg [9] on this point).

We are then led to state a principle of "super-relativity" that generalizes Einstein's wording [10], according to which the laws of nature must apply *whatever the state of the coordinate system*. This state is defined by the origin, the orientation of the axes and the motion (these 'coordinate state-variables' lead to Galileo-Poincaré-Einstein relativity), but also, in the new theory, by the resolutions. All these state variables share a common property: they can never be defined in an absolute way, but only in a relative manner. The origin, the axis angles, the velocity and acceleration (that define the *state of motion* of the coordinate system), but also the space-time resolutions, that define its *state of scale*, get physical meaning only when defined *relatively* to another system.

Therefore "super-relativity" includes motion-relativity and scale-relativity. This implies completion of the usual laws of motion and displacement of standard physics by scale laws and also new laws that couple motion and scale when non-linearity is taken into account. We recall our result [1] that the standard scale laws of the power-law self-similar type actually correspond to 'Galilean'scale laws, i.e. they have a status concerning scales similar to that of the non-relativistic laws of inertia concerning motion. From these Galilean scale laws, one can recover standard quantum mechanics as mechanics in a nondifferentiable space. The quantum behavior becomes, in this theory, a manifestation of the fractal geometry of space-time, in the same way gravitation is, in Einstein's theory of general (motion-)relativity, a manifestation of the curvature of space-time.

The evolution of the theory naturally follows that of motion relativity. One can indeed demonstrate that the standard, Galilean-like scale laws are only median-scale approximations of new laws that take a Lorentzian form when going to very small and very large space-time scales [1,3]. In these new laws, there appears a smallest and a largest resolution scale that are impassable, invariant under dilations and contractions, and replace the zero and the infinite, but keep their physical properties. These Lorentzian-like dilation laws allow one to suggest a new asymptotic behavior of quantum field theories and of cosmology. Finally, considering non-linear scale transformations, one expects the theory to be developed in the future under the form of an Einsteinian-like, general scale-relativity including fields through scale-motion coupling: such a generalized theory will be only touched upon here.

The present contribution is intended to be a review paper of the results that have been obtained in the framework of the theory of fractal space-time since the publication of Ref. [1]. It extends the contribution we have given in the first volume devoted to this subject [8], and collects new results published in Refs. [12-17]. For the coherence of reading, we again give some of the developments already described in [8], but we also complete them, including new figures and the correction of some remaining mistakes. We shall, in particular, use this occasion to collect together, for the first time in the same paper, the results and theoretical predictions concerning different domains, since the theory of scale relativity has consequences in microphysics, cosmology, and also for general chaotic systems considered at very long time scales.

2 MATHEMATICAL TOOL

2.1 Universal scale dependence on resolution

One of the main questions that is asked concerning the emergence of fractals in natural and physical sciences is the reason for their universality [5]. While particular causes may be found for their origin by a detailed description of the various systems where they appear (chaotic dynamics, biological systems, etc...) their universality nevertheless calls for a *universal* answer.

Our suggestion, which has been developed in [1, 11], is as follows. Since the time of Newton and Leibniz, the founders of the integro-differentiation calculus, one basic hypothesis which is put forward in our description of physical phenomena is that of differentiability. The strength of this hypothesis has been to allow physicists to write the equations of physics in terms of differential equations. However, there is neither a *a priori* principle nor definite experiments that impose the fundamental laws of physics to be differentiable. On the contrary, it has been shown by Feynman that typical quantum mechanical paths are non-differentiable [4].

The basic idea that underlines the theory of scale relativity is then to *give up* the arbitrary hypothesis of differentiability of space-time. In such a framework, the successes of present day differentiable physics could be understood as applying to domains where the approximation of differentiability (or integrability) was good enough, i.e. at scales such that the effects of nondifferentiability were smoothed out; but conversely, we expect the differential method to fail when confronted with truly nondifferentiable or nonintegrable phenomena, namely at very small and very large length scales (i.e., quantum physics and cosmology), and also for chaotic systems seen at very large time scales.

The new 'frontier' of physics is, in our opinion, to construct a *continuous* but *nondifferentiable* physics. (We stress the fact, well known to mathematicians, that giving up differentiability does *not* impose giving up continuity). Set in such terms, the project may seem extraordinarily difficult. Fortunately, there is a fundamental key which will be of great help in this quest, namely, the concept of scale transformations. Indeed, the main consequence of continuity and nondifferentiability is scale-divergence [1,11]. One can demonstrate that the length of a continuous and nowhere-differentiable curve is dependent on resolution ε, and, further, that $\mathcal{L}(\varepsilon) \rightarrow \infty$ when $\varepsilon \rightarrow 0$, i.e. that this curve is fractal (in a general meaning). The scale divergence of continuous and almost nowhere-differentiable curves is a direct consequence of Lebesgue's theorem, which states that *a curve of finite length is almost everywhere differentiable*.

Consider a continuous but nondifferentiable function $f(x)$ between two points $A_0[x_0, f(x_0)]$ and $A_\Omega[x_\Omega, f(x_\Omega)]$. Since f is non-differentiable, there exists a point A_1 of coordinates $[x_1, f(x_1)]$ with $x_0 < x_1 < x_\Omega$, such that A_1 is not on the segment A_0A_Ω. Then the total length $\mathcal{L}_1 = \mathcal{L}(A_0A_1) + \mathcal{L}(A_1A_\Omega) > \mathcal{L}_0 = \mathcal{L}(A_0A_\Omega)$. We can now iterate the argument and find two coordinates x_{01} and x_{11} with $x_0 < x_{01} < x_1$ and $x_1 < x_{11} < x_\Omega$, such that $\mathcal{L}_2 = \mathcal{L}(A_0A_{01}) + \mathcal{L}(A_{01}A_1) + \mathcal{L}(A_1A_{11}) + \mathcal{L}(A_{11}A_\Omega) > \mathcal{L}_1 > \mathcal{L}_0$. By iteration we finally construct successive approximations $f_0, f_1, ...f_n$ of $f(x)$ whose lengths $\mathcal{L}_0, \mathcal{L}_1, ...\mathcal{L}_n$ increase monotonically when the 'resolution' $r \approx (x_\Omega - x_0) \times 2^{-n}$ tends to zero. In other words, continuity and nondifferentiability implies a monotonous scale dependence of f (see Fig. 1).

From Lebesgue's theorem (*a curve of finite length is almost everywhere differentiable*, see Ref. [18]), one deduces that if f is continuous and almost everywhere nondifferentiable, then $\mathcal{L}(\varepsilon) \rightarrow \infty$ when the resolution $\varepsilon \rightarrow 0$, i.e., f is *scale-divergent*. This theorem is also demonstrated in Ref. [1, p.82] using non-standard analysis.

What about the reverse proposition: Is a continuous function whose length is scale-divergent between any two points (such that $x_A - x_B$ finite), i.e., $\mathcal{L}(r) \rightarrow \infty$ when $r \rightarrow 0$, non-differentiable? The answer is as follows:

If the length diverges as fast as, or faster than, a power law, i.e. $\mathcal{L}(r) \geq (\lambda/r)^\delta$, (i.e. standard fractal behavior), then the function is certainly nondifferentiable; in the intermediate domain of slower divergences (for example, logarithmic divergence, $\mathcal{L}(r) \propto \ln(\lambda/r)$, etc...), the function may be either differentiable or nondifferentiable [19]. It is interesting to see that the standard (self-similar, power-law) fractal behavior plays a critical role in this theorem: it gives the limiting behavior beyond which non-differentiability is ensured.

This result is the key for a description of nondifferentiable processes in terms of differential equations: We introduce explicitly the resolutions in the expressions

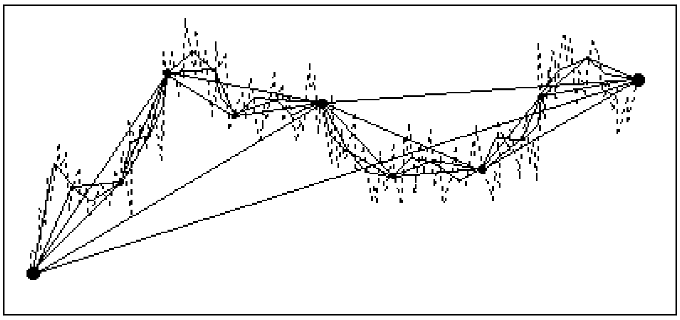

Figure 1: Construction of a non-differentiable function by successive dissections. Its length tends to infinity when the resolution interval tends to zero.

of the main physical quantities, and, as a consequence, in the fundamental equations of physics. This means that a physical quantity f, usually expressed in terms of space-time variables x, i.e., $f = f(x)$, must be now described as also depending on resolutions, $f = f(x, \varepsilon)$. In other words, rather than considering only the strictly nondifferentiable mathematical object $f(x)$, we shall consider its various approximations obtained from smoothing it or averaging it at various resolutions:

$$f(x, \varepsilon) = \int_{-\infty}^{+\infty} \Phi(x, y, \varepsilon) f(x + y) dy \tag{1}$$

where $\Phi(x, y, \varepsilon)$ is a smoothing function centered on x, for example a step function of width $\approx 2\varepsilon$, or a Gaussian of standard error $\approx \varepsilon$. (This can be also seen as a wavelet transformation, but using a filter that is not necessarily conservative). Such a point of view is particularly well adapted to applications in physics: any real measurement is always performed at finite resolution (see Refs. [1,2,11] for additional comments on this point). In this framework, $f(x)$ becomes the limit when $\varepsilon \to 0$ of the family of functions $f(x, \varepsilon)$. But while $f(x, 0)$ is nondifferentiable, $f(x, \varepsilon)$, which we have called a 'fractal function' [1], is now differentiable for all $\varepsilon \neq 0$.

The problem of the physical description of the process where the function f intervenes is now shifted. In standard differentiable physics, it amounts to finding differential equations implying the derivatives of f, namely $\partial f/\partial x, \partial^2 f/\partial x^2$, that describe the laws of displacement and motion. The integro-differentiable method amounts to performing such a local description, then integrating to get the global properties of the system under consideration. Such a method has often been called 'reductionist', and it was indeed adapted to most classical problems where no new information appears at different scales.

But the situation is completely different for systems implying fractals and non-differentiability: very small and very large scales, but also chaotic and/or turbulent systems in physics, and probably most living systems. In theses cases, new, original information exists at different scales, and the project to reduce the behavior of a system at one scale (in general, the large one) from its description at another scale (in general, the smallest, $\delta x \to 0$) seems to lose its meaning and to be hopeless. Our suggestion consists precisely to give up such a hope, and of introducing a new frame of thought where all scales co-exit simultaneously as different worlds, but are connected together via scale-differential equations.

Indeed, in non-differentiable physics, $\partial f(x)/\partial x = \partial f(x,0)/\partial x$ does not exist any longer. But the physics of the given process will be completely described if we succeed in knowing $f(x,\varepsilon)$ for all values of ε, which *is* differentiable when $\varepsilon \neq 0$, and can be the solution of differential equations involving $\partial f(x,\varepsilon)/\partial x$ but also $\partial f(x,\varepsilon)/\partial \ln\varepsilon$. More generally, if one seeks nonlinear laws, one expects the equations of physics to take the form of second order differential equations, which will then contain, in addition to the previous first derivatives, operators like $\partial^2/\partial x^2$ (laws of motion), $\partial^2/\partial(\ln\varepsilon)^2$ (laws of scale), but also $\partial^2/\partial x\partial\ln\varepsilon$, which corresponds to a coupling between motion laws and scale laws.

What is the meaning of the new differential $\partial f(x,\varepsilon)/\partial\ln\varepsilon$? This is nothing but the variation of the quantity f under an infinitesimal *scale transformation*, i.e., a dilatation of resolution. More precisely, consider the length of a nondifferentiable curve $\mathcal{L}(\varepsilon)$, and more generally a fractal curvilinear coordinate $\mathcal{L}(x,\varepsilon)$, that depends on some parameter x and on resolution ε. Such a coordinate generalizes to nondifferentiable and fractal space-time the concept of curvilinear coordinate introduced for curved, Riemannian space-time in Einstein's general relativity (see Refs. [1,2]). Let us apply an infinitesimal dilatation $\varepsilon \to \varepsilon' = \varepsilon(1+d\varrho)$ to the resolution. (We omit the x dependence in order to simplify the notation in what follows, being interested here in pure scale laws). We obtain

$$\mathcal{L}(\varepsilon') = \mathcal{L}(\varepsilon + \varepsilon d\varrho) = \mathcal{L}(\varepsilon) + \frac{\partial\mathcal{L}(\varepsilon)}{\partial\varepsilon}\varepsilon d\varrho = (1 + d\varrho\,\tilde{D})\mathcal{L}(\varepsilon) \tag{2}$$

where $\tilde{D}$ is by definition the dilatation operator. The comparison of the two last members of this equation thus yields

$$\tilde{D} = \varepsilon\frac{\partial}{\partial\varepsilon} = \frac{\partial}{\partial\ln\varepsilon}. \tag{3}$$

This well known form of the infinitesimal dilatation operator, obtained by an application to this problem of the Gell-Mann-Levy method (see e.g. Ref. [20]) shows that the 'natural' variable for resolution is $\ln\varepsilon$, and that the expected new differential equations will indeed involve quantities like $\partial\mathcal{L}(x,\varepsilon)/\partial\ln\varepsilon$. Now equations describing the scale dependence of physical beings have already been introduced in physics: these are the renormalization group equations, particularly developed in the framework of Wilson's 'multiple-scale-of-length' approach [21]. In its simplest form, a renormalization-group-like equation for an essential physical quantity like $\mathcal{L}$ can be interpreted as stating that the variation of $\mathcal{L}$ under an infinitesimal scale transformation $d\ln\varepsilon$ depends only on $\mathcal{L}$ itself (namely, $\mathcal{L}$ determines the whole physical behavior, including the behavior in scale transformations). This reads:

$$\frac{\partial\mathcal{L}(x,\varepsilon)}{\partial\ln\varepsilon} = \beta(\mathcal{L}). \tag{4}$$

2.2 Galilean scale-relativity

Once again looking for the simplest possible form for such an equation, we expand $\beta(\mathcal{L})$ in powers of $\mathcal{L}$. This can be done since one may always renormalize $\mathcal{L}$ dividing it by some large value $\mathcal{L}_1$, ($\mathcal{L} \to \mathcal{L}' = \mathcal{L}/\mathcal{L}_1$) in such a way that the new variable $\mathcal{L}'$ remains $<< 1$ in the domain of interest. We obtain to first order the linear equation

$$\frac{\partial \mathcal{L}(x,\varepsilon)}{\partial \ln \varepsilon} = a + b\mathcal{L}. \tag{5}$$

Such an expansion can always been done, since one may renormalize $\mathcal{L}$ dividing it by its largest value in its domain of variation. The solution of equation (5) is

$$\mathcal{L}(x,\varepsilon) = \mathcal{L}_o(x) \left[1 + \zeta(x) \left(\frac{\lambda}{\varepsilon}\right)^{-b}\right], \tag{6}$$

where $\lambda^{-b}\zeta(x)$ is an integration 'constant' and $\mathcal{L}_o = -a/b$.

These notations allow us to chose $\zeta(x)$ such that $<\zeta^2(x)>= 1$. Provided $a \neq 0$, equation (6) clearly shows two domains. Assume first $b < 0$:

(i) $\varepsilon << \lambda$: in this case $\zeta(x)(\lambda/\varepsilon)^{-b} >> 1$, and $\mathcal{L}$ is given by a scale-invariant fractal-like power law with fractal dimension $D = 1 - b$, namely $\mathcal{L}(x,\varepsilon) = \mathcal{L}_o(x)(\lambda/\varepsilon)^b$.

(ii) $\varepsilon >> \lambda$: then $\zeta(x)(\lambda/\varepsilon)^{-b} << 1$, and $\mathcal{L}$ becomes independent of scale.

We stress the fact that Eq.(6) gives us not only a fractal (scale-invariant) behavior at small scale, but also a transition from fractal to nonfractal behavior at scales larger than some transition scale λ. In other words, our generalization of renormalization group-like equations in its simplest linear form (which includes a zeroth order term $\beta(0)$) is able to provide us not only with scale invariance, but also with the spontaneous breaking of this fundamental symmetry of nature. Only the particular case $a = 0$ yields unbroken scale invariance, $\mathcal{L} = \mathcal{L}_o(\lambda/r)^\delta$, where $\delta = -b$ is a 'scale dimension' [20]. Note that the corresponding equation (4) in this case writes:

$$\tilde{D}\mathcal{L} = b\mathcal{L}, \tag{7}$$

i.e. the scale dimension is given by the eigenvalue of the dilatation operator.

The solutions corresponding to the case $b > 0$ are the symmetric of the case $b < 0$ (see Fig. 2). The scale dependence is at large scales and is broken to yield scale independence below the transition λ. While $b < 0$ is characteristic of the microphysical situation (it yields both quantum phenomena - Schrödinger's equation and correspondence principle- and the quantum-classical transition [1,11] as we shall see below), the case $b > 0$ is also of profound physical significance, since it is encountered in the cosmological domain [1,16].

We think that the above mechanism is the clue to understanding the universality of fractals in nature. Self-similar, scale-invariant fractals with constant fractal dimension are nothing but the simplest possible behaviour of nondifferentiable, scale-dependent phenomena. They correspond to the *linear case* of scale laws, the equivalent of which are inertial frames for motion laws (this analogy will be reinforced in what follows). The advantage of such an interpretation is that it opens several roads for generalization, the most promising being to implement the principle of scale relativity thanks to a generalization of scale invariance, namely, scale *covariance* of the equations of physics, as we shall now see [1,3].

Such a generalization toward scale covariance leads one to attribute a new physical meaning to the fractal dimension: in scale relativity, it becomes the component of a vector, and so varies explicitly in function of resolution and plays for scale laws a role similar to that played by time for motion laws [1,3,11].

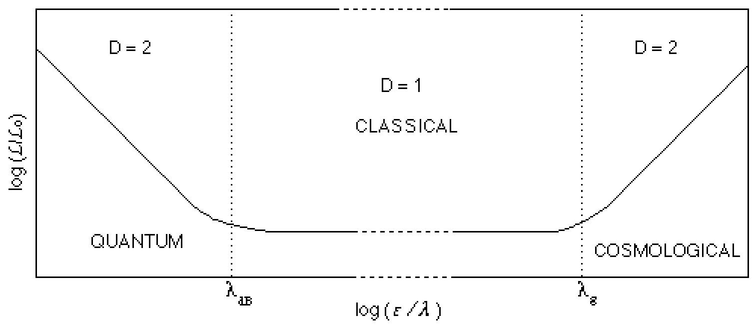

Figure 2: Typical behavior of the solutions to the simplest linear scale differential equation. One obtains an asymptotic fractal (power-law resolution-dependent) behavior at either large or small scales, and a transition to scale-independence toward the classical domain (intermediate scales).

2.3 Special scale relativity

The question that we shall now address is that of finding the laws of scale transformations that meet the principle of scale relativity. Up to now, we have characterized typical scale laws as the simplest possible laws, namely, those which are solutions of the simplest form of linear scale differential equations: this reasoning has provided us with the standard, power-law, fractal behavior with constant fractal dimension in the asymptotic domain. But are the simplest possible laws those chosen by nature? Experience in the construction of the former physical theories suggests that the correct and general laws are simplest among those which satisfy some fundamental principle, rather than those which are written in the simplest way: anyway, these last laws are often approximations of the correct, more genral laws. Good examples of such relations between theories are given by Einstein's motion special relativity, of which the Galilean laws of inertial motion are low velocity approximations, and by Einstein's general relativity, which includes Newton's theory of gravitation as an approximation. In both cases, the correct laws are constructed from the requirement of covariance, rather than from the too simple requirement of invariance.

The theory of scale relativity [1,3] proceeds along a similar reasoning. The principle of scale relativity may be implemented by requiring that the equations of physics be written in a covariant way under scale transformations of resolutions. Are the standard scale laws (those described by renormalization-group-like equations, or by a fractal power-law behavior) scale-covariant ? They are usually described (far from the transition to scale-independence) by asymptotic laws such as $\varphi = \varphi_o(\lambda/r)^\delta$, with δ a *constant* scale-dimension (which may differ from the standard value $\delta = 1$ by an anomalous dimension term [20]). This means that a scale transformation $r \to r'$ can be written:

$$\ln \frac{\varphi(r')}{\varphi_o} = \ln \frac{\varphi(r)}{\varphi_o} + \mathbb{V}\ \delta(r), \tag{8}$$

$$\delta(r') = \delta(r),$$

where we have set:

$$\mathbb{V} = \ln(r/r'). \tag{9}$$

The choice of a logarithmic form for the writing of the scale transformation and the definition of the fundamental resolution parameter $\mathbb{V}$ is justified by the expression of the dilatation operator $\tilde{D} = \partial/\partial \ln r$ (Eq. 3). The relative character of $\mathbb{V}$ is evident: in the same way that only velocity differences have a physical meaning (Galilean relativity of motion), only $\mathbb{V}$ differences have a physical meaning (relativity of scales). We have then suggested [3] to characterize this relative resolution parameter $\mathbb{V}$ as a 'state of scale' of the coordinate system, in analogy with Einstein's formulation of the principle of relativity [10], in which the relative velocity characterizes the state of motion of the reference system.

Now in such a frame of thought, the problem of finding the laws of linear transformation of fields in a scale transformation $r \to r'$ amounts to finding four quantities, $a(\mathbb{V}), b(\mathbb{V}), c(\mathbb{V})$, and $d(\mathbb{V})$, such that

$$\ln \frac{\varphi(r')}{\varphi_o} = a(\mathbb{V}) \ln \frac{\varphi(r)}{\varphi_o} + b(\mathbb{V})\, \delta(r), \tag{10}$$

$$\delta(r') = c(\mathbb{V}) \ln \frac{\varphi(r)}{\varphi_o} + d(\mathbb{V})\, \delta(r).$$

Set in this way, it immediately appears that the current 'scale-invariant'scale transformation law of the standard renormalization group (Eq. 8), given by $a = 1, b = \mathbb{V}, c = 0$ and $d = 1$, corresponds to the Galileo group.

This is also clear from the law of composition of dilatations, $r \to r' \to r"$, which has a simple additive form,

$$\mathbb{V}" = \mathbb{V} + \mathbb{V}'. \tag{11}$$

However the general solution to the 'special relativity problem' (namely, find a, b, c and d from the principle of relativity) is the Lorentz group [22,3,1]. Then we have suggested [3] to replace the standard law of dilatation, $r \to r' = \varrho r$ by a new Lorentzian relation. However, while the relativistic symmetry is universal in the case of the laws of motion, this is not true for the laws of scale. Indeed, physical laws are no longer dependent on resolution for scales larger than the classical-quantum transition (identified with the fractal-nonfractal transition in our approach) which has been analysed above. This implies that the dilatation law must remain Galilean above this transition scale.

For simplicity, we shall consider in what follows only the one-dimensional case. We define the resolution as $r = \delta x = c\delta t$, and we set $\lambda_0 = c\tau_{dB} = \hbar c/E$. In its rest frame, λ_0 is thus the Compton length of the system or particle considered, i.e., in the first place the Compton length of the electron (this will be better justified in Section 6). Our new law of dilatation reads, for $r < \lambda_0$ and $r' < \lambda_0$

$$\ln \frac{r'}{\lambda_0} = \frac{\ln(r/\lambda_0) + \ln \varrho}{1 + \frac{\ln \varrho \, \ln(r/\lambda_0)}{\ln^2(\lambda_0/\mathbf{\Lambda})}}. \tag{12}$$

This relation introduces a fundamental length scale $\mathbf{\Lambda}$, that we have identified with the Planck length (currently $1.61605(10) \times 10^{-35}$ m),

$$\mathbf{\Lambda} = (\hbar G/c^3)^{1/2}. \tag{13}$$

But, as one can see from Eq.(12), if one starts from the scale $r = \mathbf{\Lambda}$ and apply any dilatation or contraction ϱ, one gets back the scale $r' = \mathbf{\Lambda}$, whatever the

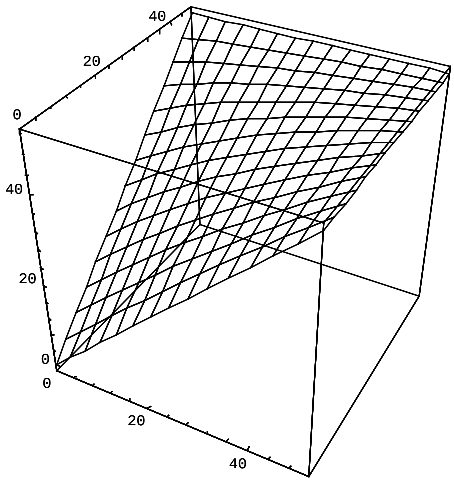

Figure 3: Illustration of the new law of composition of dilations in special scale relativity (Eq. 12). The new product $\varrho" = \varrho \times \varrho'$ takes a Lorentzian form (in log) and so remains limited. Here we have taken $\mathbb{C} = \ln(\lambda_e/\mathbf{\Lambda}) \approx 51.528$.

initial value of λ_0 (i.e., whatever the state of motion, since λ_0 is Lorentz-covariant under *velocity* transformations). In other words, $\mathbf{\Lambda}$ is now interpreted as a limiting lower length-scale, impassable, invariant under dilatations and contractions. The length measured along a fractal coordinate, that was previously scale-dependent as $\ln(\mathcal{L}/\mathcal{L}_0) = \delta_0 \ln(\lambda_0/r)$ for $r < \lambda_0$ becomes in the new framework (in the simplified case of pure scale laws: see [1,3] for general expressions)

$$\ln(\mathcal{L}/\mathcal{L}_0) = \frac{\delta_0 \ln(\lambda_0/r)}{\sqrt{1 - \ln^2(\lambda_0/r)/\ln^2(\lambda_0/\mathbf{\Lambda})}}. \tag{14}$$

The main new feature of scale relativity respectively to the previous fractal or scale-invariant approaches is that the scale dimension δ and the fractal dimension $D = 1 + \delta$, which were previously constant ($D = 2, \delta = 1$), are now explicitly varying with scale, following the law (here once again simplified by neglecting the small dependence on length):

$$\delta(r) = \frac{1}{\sqrt{1 - \ln^2(\lambda_0/r)/\ln^2(\lambda_0/\mathbf{\Lambda})}}. \tag{15}$$

This means that the fractal dimension, which jumps from $D = 1$ to $D = 2$ at the electron Compton scale $\lambda_0 = \lambda_e = \hbar/m_e c$ [1,2], is now varying with scale, at first very slowly as

$$D(r) = 2(1 + \frac{1}{4}\frac{\mathbb{V}^2}{C_0^2} + ...), \tag{16}$$

where $\mathbb{V} = \ln(\lambda_0/r)$ and $C_0 = \ln(\lambda_0/\mathbf{\Lambda})$, then tends to infinity at very small scales when $\mathbb{V} \to C_0$, i.e., $r \to \mathbf{\Lambda}$. When λ_0 is the Compton length of the electron, the new fundamental constant C_0 is found to be

$$C_e = \ln(\frac{m_P}{m_e}) = 51.52797(7) \tag{17}$$

from the experimental values of the electron and Planck masses [23] (the number into brackets is the uncertainty on the last digits).

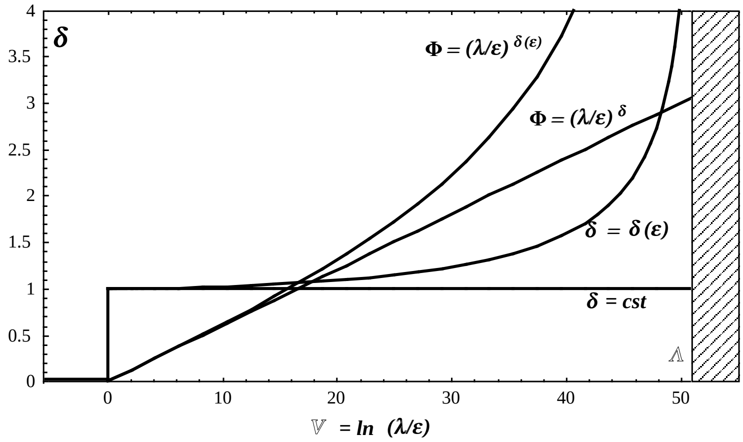

Figure 4: Illustration of the passage from Galilean scale relativity to special (Lorentzian) scale relativity. The fractal dimension, that was constant at small scale in "Galilean" laws, becomes scale dependent and now diverges when resolution tends to the Planck scale, that now owns all the physical properties of the zero scale.

Let us sum up various new formulas of special scale-relativity (right-hand side formulas below) and compare them to the former standard laws (left-hand side formulas), that correspond to the "Galilean" version of scale relativity (see Refs. [1,3,11,12] and Fig. 4).

Scale dependence of coupling constants:

$$\alpha^{-1}(r) = \alpha^{-1}(\lambda_0) + \beta_0 \ln\frac{\lambda_0}{r} + ...; \text{unchanged in terms of length} - \text{scale (to first order)} \tag{18}$$

Compton relation:

$$\ln\frac{m}{m_0} = \ln\frac{\lambda_0}{\lambda}, \quad \text{with } \lambda_0 m_0 = \hbar/c; \quad \ln\frac{m}{m_0} = \frac{\ln(\lambda_0/\lambda)}{\sqrt{1-\frac{\ln^2(\lambda_0/\lambda)}{\ln^2(\lambda_0/\mathbf{\Lambda})}}}. \tag{19}$$

Heisenberg's inequality:

$$\ln\frac{\sigma_p}{p_0} \geq \ln\frac{\lambda_0}{\sigma_x}, \text{ with } \lambda_0 p_0 = \hbar; \quad \ln\frac{\sigma_p}{p_0} \geq \frac{\ln(\lambda_0/\sigma_x)}{\sqrt{1-\frac{\ln^2(\lambda_0/\sigma_x)}{\ln^2(\lambda_0/\mathbf{\Lambda})}}}. \tag{20}$$

Scale invariant:

$$\delta = \text{constant}; \quad d\sigma^2 = C_0^2\, d\delta^2 - \frac{d\mathcal{L}^2}{\mathcal{L}^2}, \quad \text{with } C_0 = \ln(\frac{\lambda_0}{\mathbf{\Lambda}}) \tag{21}$$

In these equations, valid for $r \leq \lambda_0$, λ_0 is the Compton length of the electron (or its Lorentz transform), and there appears a fundamental length-scale $\mathbf{\Lambda}$, that is invariant under dilations and contractions (in a way similar to the invariance of the velocity of light under motion transformations), and which can be naturally identified with the Planck length, $(\hbar G/c^3)^{1/2}$, (see [1] for more detail).

As we shall see in the next section, one can describe the elementary displacements dX^i on a geodesic of a fractal space-time in terms of a mean $dx^i = < dX^i >$ and a fluctuation $d\xi^i = dX^i - dx^i$. The fact that the fluctuations $d\xi^i$ have positive definite metric (even in classical Minkowski 4-space-time) allows us to implement special scale-relativity from the introduction, at small space-time scales $\varepsilon < \lambda_0$, of a 5-dimensional Minkowskian "space-time-zoom" of signature $(+, -, -, -, -)$, whose fifth dimension is the (now variable) "scale" dimension δ [12]:

$$d\sigma^2 = C_0^2 d\delta^2 - \Sigma_i \frac{(d\xi^i)^2}{\xi^2}, \tag{22}$$

where $C_0 = \ln(\lambda_0/\mathbf{\Lambda})$.

This is to be compared to the achievement of Einstein-Poincaré special motion-relativity thanks to the introduction of Minkowskian space-time: motion in space can be described as rotation in space-time, and the full Lorentz invariance is finally implemented by jumping to a 4-dimensional tensor description. Here the situation is similar, with δ playing for scale the same role as played by time for motion. There too, the scale transformations, i.e., the dilatations and contractions of space-time resolutions, are reduced to rotations in our 5-dimensional Minkowskian fractal space-time.

2.4 Generalized scale relativity: first hints

Note however that, while the simplified case of global scale dilatations corresponds to a Minkowskian (1+1) space (Eq. 21), the general situation where we allow 4 different dilatations of the four space-time resolutions implies a form of the metric that already comes under a theory of "general scale relativity". The full development of such a generalized theory, that is expected to include nonlinear scale transformations in terms of second order scale differential equations, lies outside the scope of the present contribution. We can nevertheless already give some hints about the fundamental structures of such a generalized future theory. We expect Eq.(22) to remain only a "local" relation in this generalized theory, following a principle of "scale-equivalence" according to which there always exists locally a "scale-inertial" coordinate system in which the invariant keeps the Minkowskian form (Eq. 22). The state of scale of the coordinate system in which Eq.(22) applies is the equivalent for resolutions of "free fall" motion. We know that free fall motion is, in the simplest case, uniformly accelerated motion. In the same way, we expect the "free fall state of scale" to imply second derivatives like $d^2 \ln \xi / d\delta^2$, i.e., scale "accelerations" $d\mathbb{V}/d\delta$. It is highly probable that most of today's experimental devices, being in general characterized by a fixed, invariant resolution (example: energy of an accelerator), are unable to put into evidence such a state of scale. These ideas may then lead in the end to the concept of completely new kind of experiments in microphysics.

In a more general coordinate system, the scale invariant is expected to take an Einsteinian metric form:

$$d\sigma^2 = G_{ij} d\xi^i d\xi^j , \tag{23}$$

where i and j now run form 0 to 4, and where $d\xi^0$ is identified with the scale dimension or "zoom", δ. The G_{ij} 's represent the potentials of a new "scale field", or, in other words, the general scale structure of the fractal space-time. We think that such a field can be related to the nature of the charges of the gauge interactions (see Sec. 5), and, more profoundly, to the renormalization group equation, i.e., to the Callan-Symanzik β functions. We expect its variation to be given, in a way that will be specified in a forthcoming work, by equations similar to Einstein's field equations of general relativity.

The classical four-dimensional space-time can then "emerge" from this five-dimensional space-time-zoom [13]. Indeed, when going back to large space-time scales: (i) the scale dimension jumps to a constant value, then $d\delta = 0$ and the five-space degenerates into a four-space; (ii) among the 4 variables X^i, we can singularize the one which is characterized by the smallest fractal-nonfractal transition, say X^1, and call it "time" ; then one can demonstrate [Ref. 1, pp.123-4] that the fractal divergences cancel in such a way that the three quantities $< dX^i/dX^1 >, i = 2, 3, 4,$ remain smaller than 1 (i.e., smaller than c in units m/s); (iii) this implies Einstein's motion-relativity, i.e., equivalently, a Minkowskian metric for the mean (classical) remaining four variables.

Let us conclude this section by remarking that, at very high energy, when approaching the Planck mass scale, the physical description is reduced to that of the "quantum" variables ξ^i, while the average, classical variables x^i lose their physical meaning: we recover by another way the fact that gravitation and quantum effects become of the same order, and that a theory of quantum gravity is needed. However, the variables with which such a theory must be constructed are the ξ^i 's rather than the x^i 's. The fact that they are positive-definite instead of being of signature $(+, -, -, -)$ justifies the need to work in the framework of an Euclidean rather than Minkowskian space-time when performing path integral calculations in quantum gravity. However, if one wants to include our new interpretation of the Planck length-scale and to build a special scale relativistic theory, one must jump again to a hyperbolic signature in five dimensions.

3 FRACTAL SPACE-TIME AND GEODESICS

We shall now try to be more specific about what we mean by the concept of "fractal space-time" and that of its geodesics, then about their mathematical description [15]. The only presently existing "space-time theory" is Einstein's relativity theory, first special (involving an absolute Minkowskian space-time), then general (relative Riemannian space-time). Clearly the development of a fractal space-time theory is far from having reached the same level of elaboration. However, at the simple level of the theory that we consider here (Schrödinger and Klein-Gordon equations, then their scale-relativistic Lorentzian generalizations) we can already identify the various elements of the description that are specific of a space-time theory. These elements are:

(i) A description of the laws that govern the elementary space-time displacements, including in particular the quantities that remain invariant in transformations of coordinates. In general relativity, the most important is the metric $ds^2 = g_{ij} dx^i dx^j$, that contains the gravitational potentials g_{ij}. In the case of a non-differentiable, fractal space-time, an enlarged group of transformations must be considered, that includes resolution transformations.

(ii) A description of the effects of displacements on other physical quantities (completed in our generalization by the effects of scale transformations on these quantities). The power of the space-time / relativity approach is that all these effects can be calculated using a unique mathematical tool, the *covariant derivative.* This covariant derivative depends on the geometry of space-time. In general relativity, it writes $D_i A^j = \partial_i A^j + \Gamma^j_{ik} A^k$: the geometry is described by the Christoffel symbols Γ^j_{ik}, (the "gravitational field"). Covariance implies that the *equations of geodesics* take the simplest possible form, that of free motion, $D^2 x^j / ds^2 = 0$, i.e., the covariant acceleration vanishes. (Recall that in general relativity it is developed as $d^2 x^i / ds^2 + \Gamma^i_{jk} (dx^j / ds)(dx^k / ds) = 0$, which generalizes Newton's equation, $m \, d^2 x^i / dt^2 = F^i$).

(iii) Equations constraining the geometries that are physically acceptable, relating them to their material-energetic content (in general relativity, Einstein's equations).

Let us now show how we have constructed the equivalent of these tools in the case of a nondifferentiable, fractal geometry. We shall only sum up here the great lines of the construction: more complete demonstrations can be found in Refs. [1, 11-16]. Consider a small increment dX^i of nondifferentiable 4-coordinates along one of the geodesics. We can decompose dX^i in terms of its mean, $< dX^i >= dx^i$ and a fluctuation respective to the mean, $d\xi^i$ (such that $< d\xi^i >= 0$ by definition):

$$dX^i = dx^i + d\xi^i. \tag{24}$$

From their very definition, the variables dx^i will generalize the classical variables, while the $d\xi^i$ describe the new, non-classical, fractal behavior. According to Section 2 and to the now well-known laws of fractal geometry [5,24] (see Fig. 5), this behavior depends on the fractal dimension D as:

$$d\xi \propto dt^{1/D}. \tag{25}$$

We shall first consider the nonrelativistic case (Schrödinger's equation), so that the motion invariant is now the standard time coordinate. Let us introduce classical and "fractal" velocities v^i and u^i, such that $< (u^i)^2 >= 1$ and $< u^i >= 0$. In the particular case $D = 2$, which is a critical value in our approach (see [1,25]), we may write Eq.(24) as:

$$dX^i = v^i dt + \lambda^{1/2} \, u^i \, dt^{1/2}. \tag{26}$$

A second step in the construction of our mathematical tool consists in realizing that, because of nondifferentiability, there will be an infinity of fractal geodesics between any couple of points in a fractal space-time. In a theory based on these concepts, we can make predictions only using the infinite family of geodesics. These predictions so become of a probabilistic nature, though particles can be identified with one particular (but undetermined) geodesic of the family [1,11]. This forces one to jump to a statistical description. Equation (25) means that, when $D = 2$, the fluctuation is such that $< d\xi^2 > \propto dt$, and one recognizes here the basic law of a Markov-Wiener process.

The third step provides us with an explanation of the need to use complex numbers in quantum mechanics. Indeed, the nondifferentiable nature of space-time implies a breaking of time reflection invariance. In other words, the elementary process described in Eq. (24) is *fundamentally irreversible*, since the mean forward derivative of a given function is *a priori* different from the mean backward one (see Fig. 6). But the choice of a reversed time coordinate ($-t$) must be as qualified as the initial choice (t) for the description of the laws of nature. The only solution to this problem amounts to considering both the forward ($dt > 0$) and backward ($dt < 0$) processes "simultaneously". Information is doubled with respect to the classical, differentiable description. This doubling of information can be accounted for using complex numbers and the complex product, but one can demonstrate that this is a particular choice of representation, that achieves the simplest description (i.e., using a different product would introduce new terms in the Schrödinger equation [19]). The new complex process, *as a whole*, recovers the fundamental property of microscopic reversibility.

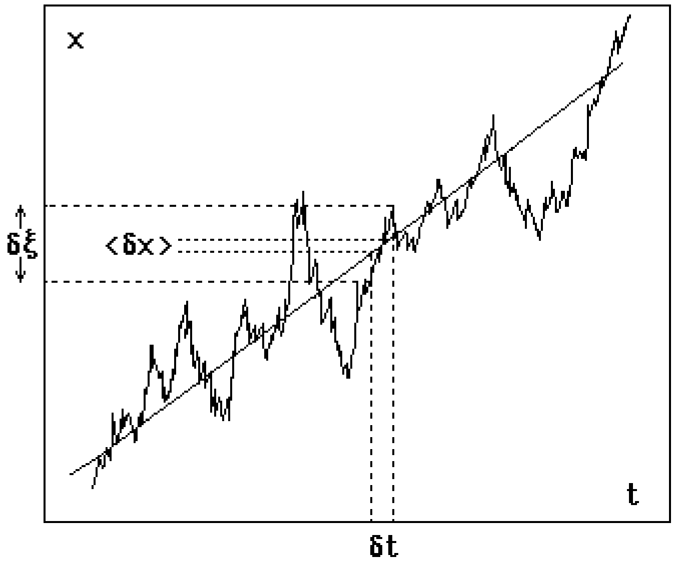

Figure 5: Relation between space and time differential elements, in the case of a fractal function. While the average, "classical" coordinate variation $\delta x =< \delta X >$ is of the same order as the time differential δt , the fluctuation becomes much larger than δt when $\delta t << \tau$, and depends on the fractal dimension D as: $\delta\xi \propto \delta t^{1/D}$.

We are now able to sum up the minimum ingredients that describe elementary displacements in a fractal space-time. There is a first doubling of variables, in terms of mean displacements, dx (that generalizes the classical variables) and fluctuation, $d\xi$. Then a second doubling occurs because of the time reversibility at the elementary level. We must then write instead of equation (24):

$$dX^i_\pm = dx^i_\pm + d\xi^i_\pm, \tag{27}$$

for the two forward (+) and backward (−) processes. The new explicit dependence on scale (i.e., here, time resolution) is accounted for by the description of the fluctuations. The fundamental fractal relation (equation 25 and Figure 5) is generalized to 3 dimensions as:

$$< \frac{d\xi^i_\pm}{dt} \frac{d\xi^j_\pm}{dt} >= \pm \delta^{ij} \left(\frac{\lambda}{\delta t} \right)^{2-2/D} . \tag{28}$$

This relation is invariant under translations and rotations in space between Cartesian coordinate systems. In the special, "Galilean" case that we consider for the moment, the scale invariant is the fractal dimension itself, $D = 1 + \delta$, with:

$$\delta = 1 = \text{invariant.} \tag{29}$$

Then the $d\xi^i_\pm(t)$ correspond to a Markov-Wiener process, i.e., they are Gaussian with mean zero, mutually independent and such that

$$< d\xi^i_\pm d\xi^j_\pm >= \pm\lambda\, \delta^{ij} dt. \tag{30}$$

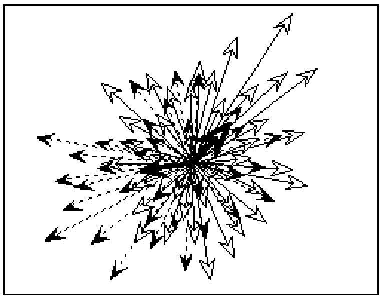

Figure 6: Illustration of the doubling of the mean velocity due to non-differentiability. There are an infinity of forward geodesics (increasing time) passing by a given point, and an infinity of backward ones (after time reversal). Even after averaging, the mean forward and backward velocities are a priori different.

We can now jump to the second step of the space-time description, by constructing the *covariant derivative* that describes the effects of the new displacement and scale laws. We define, following Nelson [26,27], mean forward (+) and backward (−) derivatives,

$$\frac{d_\pm}{dt} y(t) = \lim_{\delta t \to 0\pm} < \frac{y(t+\delta t) - y(t)}{\delta t} >, \tag{31}$$

which, once applied to x^i, yield *forward and backward mean velocities*, $d_+ x^i(t)/dt = v^i_+$ and $d_- x^i(t)/dt = v^i_-$. The forward and backward derivatives of equation (31) can be combined in terms of a *complex* derivative operator [1,11],

$$\frac{\acute{d}}{dt} = \frac{(d_+ + d_-) - i(d_+ - d_-)}{2dt}, \tag{32}$$

which, when applied to the position vector, yields a complex velocity

$$\mathcal{V}^i = \frac{\acute{d}}{dt} x^i = V^i - i\, U^i = \frac{v^i_+ + v^i_-}{2} - i\, \frac{v^i_+ - v^i_-}{2}. \tag{33}$$

The real part V^i of the complex velocity $\mathcal{V}^i$ generalizes the classical velocity, while its imaginary part, U^i, is a new quantity arising from the non-differentiability. Consider a function $f(X,t)$, and expand its total differential to second order. We get

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \nabla f . \frac{dX}{dt} + \frac{1}{2} \frac{\partial^2 f}{\partial X_i \partial X_j} \frac{dX_i dX_j}{dt}. \tag{34}$$

We may now compute the mean forward and backward derivatives of f. In this procedure, the mean value of dX/dt amounts to $d_\pm x/dt = v_\pm$, while $< dX_i dX_j >$ reduces to $< d\xi_{\pm i} d\xi_{\pm j} >$, so that the last term of equation (34) amounts to a Laplacian thanks to equation (30). We obtain

$$d_\pm f/dt = (\partial/\partial t + v_\pm.\nabla \pm \frac{1}{2}\lambda\Delta)f. \tag{35}$$

By combining them we get our final expression for our complex scale-covariant derivative:

$$\frac{\acute{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\frac{1}{2}\lambda\Delta. \tag{36}$$

We shall now apply the *principle of scale covariance*, and postulate that the passage from classical (differentiable) mechanics to the new nondifferentiable mechanics that is considered here can be implemented by a unique prescription: Replace the standard time derivative d/dt by the new complex operator $\acute{d}/dt$. As a consequence, we are now able to write the equation of the geodesics of the fractal space-time under their covariant form:

$$\frac{\acute{d}^2}{dt^2} x^i = 0. \tag{37}$$

As we shall recall hereafter (and as already demonstrated in Refs. [1,11-15]), this equation amounts to the free particle Schrödinger equation.

The last step in our construction consists in writing the “field” equations, i.e., the equations that relate the geometry of space-time to its matter-energy content. At the level that is considered here, there is only one geometrical free parameter left in the expression of the covariant derivative, namely the length-scale λ, that appeared as an integration constant in equations (6) and (26). In order to recover quantum mechanics, λ must be the Compton length of the particle considered, i.e., $\lambda = \hbar/m$: it takes the new geometrical meaning of a fractal / non-fractal transition (in the dimension of resolution).

4 SCALE COVARIANCE AND QUANTUM MECHANICS

4.1 Schrödinger equation

Let us now recall the main steps by which one may pass from classical mechanics to quantum mechanics using our scale-covariance [1,11-15]. We assume that any mechanical system can be characterized by a Lagrange function $\mathcal{L}(x, \mathcal{V}, t)$, from which an action $\mathcal{S}$ is defined:

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}(x, \mathcal{V}, t) dt. \tag{38}$$

Our Lagrange function and action are *a priori* complex since $\mathcal{V}$ is complex, and are obtained from the classical Lagrange function $L(x, dx/dt, t)$ and classical action S precisely by applying the above prescription $d/dt \to \acute{d}/dt$. The principle of stationary action, $\delta\mathcal{S} = 0$, applied to this new action with both ends of the above integral fixed, leads to generalized Euler-Lagrange equations [1]:

$$\frac{\acute{d}}{dt}\frac{\partial\mathcal{L}}{\partial\mathcal{V}_i} = \frac{\partial\mathcal{L}}{\partial x_i}. \tag{39}$$

Other fundamental results of classical mechanics are also generalized in the same way. In particular, assuming homogeneity of space in the mean leads to defining a generalized *complex* momentum given by

$$\mathcal{P} = \frac{\partial\mathcal{L}}{\partial\mathcal{V}}. \tag{40}$$

If one now considers the action as a functional of the upper limit of integration in equation (38), the variation of the action yields another expression for the complex momentum, as well as a generalized complex energy:

$$\mathcal{P} = \nabla\mathcal{S} \ \ ; \ \ \mathcal{E} = -\partial\mathcal{S}/\partial t. \tag{41}$$

We now specialize and consider Newtonian mechanics. The Lagrange function of a closed system, $L = \frac{1}{2}mv^2 - \Phi$, is generalized as $\mathcal{L}(x, \mathcal{V}, t) = \frac{1}{2}m\mathcal{V}^2 - \Phi$, where Φ denotes a (still classical) scalar potential. The Euler-Lagrange equation keeps the form of Newton's fundamental equation of dynamics

$$m\,\frac{\acute{d}}{dt}\mathcal{V} = -\nabla\Phi, \tag{42}$$

but is now written in terms of complex variables and operator. In the free particle case ($\Phi = 0$), we recover the geodesics equation (37), $\acute{d}^{\,2}x^i/dt^2 = 0$. The complex momentum $\mathcal{P}$ now reads:

$$\mathcal{P} = m\mathcal{V}, \tag{43}$$

so that from equation (41) we arrive at the conclusion that the *complex velocity* $\mathcal{V}$ *is a gradient*, namely the gradient of the complex action:

$$\mathcal{V} = \nabla\mathcal{S}/m. \tag{44}$$

We may now introduce a complex function ψ which is nothing but another expression for the complex action $\mathcal{S}$,

$$\psi = e^{i\mathcal{S}/m\lambda}. \tag{45}$$

It is related to the complex velocity as follows:

$$\mathcal{V} = -i\lambda\nabla(\ln\psi). \tag{46}$$

From this equation and equation (43), we obtain:

$$\mathcal{P}\psi = -im\lambda\nabla\psi, \tag{47}$$

which is nothing but the *correspondence principle* of quantum mechanics for momentum (since $m\lambda = \hbar$), but here demonstrated and written in terms of an exact equation. The same is true for the energy: $\mathcal{E} = -\partial \mathcal{S}/\partial t = im\lambda \partial(\ln \psi)/\partial t \Rightarrow \mathcal{E}\psi = i\hbar \partial \psi / \partial t$. We have now at our disposal all the mathematical tools needed to write the complex Newton equation (42) in terms of the new quantity ψ. It takes the form

$$\nabla \Phi = im\lambda \frac{\acute{d}}{dt}(\nabla \ln \psi). \tag{48}$$

Replacing $\acute{d}/dt$ by its expression (36) yields:

$$\nabla \Phi = i\lambda m[\frac{\partial}{\partial t}\nabla \ln \psi - i\frac{\lambda}{2}\Delta(\nabla \ln \psi) - i\lambda(\nabla \ln \psi.\nabla)(\nabla \ln \psi)]. \tag{49}$$

Standard calculations allows one to simplify this expression thanks to the relation

$$\frac{1}{2}\Delta(\nabla \ln \psi) + (\nabla \ln \psi.\nabla)(\nabla \ln \psi) = \frac{1}{2}\nabla \frac{\Delta \psi}{\psi}, \tag{50}$$

and we obtain

$$\frac{\acute{d}}{dt}\mathcal{V} = -\lambda \nabla[i\frac{\partial}{\partial t}\ln \psi + \frac{\lambda}{2}\frac{\Delta \psi}{\psi}] = -\nabla \Phi / m. \tag{51}$$

Integrating this equation yields

$$\frac{1}{2}\lambda^2 \Delta \psi + i\lambda \frac{\partial}{\partial t}\psi - \frac{\Phi}{m}\psi = 0, \tag{52}$$

up to an arbitrary phase factor $\alpha(t)$ which may be set to zero by a suitable choice of the phase of ψ. We finally get the standard form of Schrödinger's equation, since $\lambda = \hbar/m$:

$$\frac{\hbar^2}{2m}\Delta \psi + i\hbar \frac{\partial}{\partial t}\psi = \Phi \psi. \tag{53}$$

In our approach, we have obtained the Schrödinger equation introducing neither the probability density, nor the Kolmogorov equations (contrary to Nelson's [26,27] stochastic mechanics, in which the Schrödinger equation is a pasting of the real Newton equation and of Fokker-Planck equations for a Brownian diffusion process). Hence this theory is not statistical in its essence (nor is it a diffusion theory), but must be completed, as has been historically the case for quantum mechanics itself, by a statistical interpretation. It is simply obtained by setting $\psi\psi^\dagger = \varrho$. Now writing the imaginary part of equation (53) in terms of this new variable, one gets the equation of continuity:

$$\partial \varrho / \partial t + div(\varrho V) = 0, \tag{54}$$

so that ϱ is easily identified with a probability density.

4.2 Quantum Mechanics of Many Identical Particles

We have shown in Ref. [15] that our theory was also able to provide us with the Schrödinger equation in the case of many identical particles: let us recall this result.

The many-particle quantum theory plays a particularly important role in the understanding of quantum phenomena, since it is often considered as a definite proof that one cannot understand quantum mechanics in terms of a pure space-time description. Indeed the wave function must be defined in a configuration space with $3n+1$ dimension, $\psi = \psi(x_1, x_2, ...x_n; t)$, in which the variables are *a priori* not separated, even though they are separated in the Hamiltonian, $H = \Sigma_i H(x_i, p_i)$. This case is also important since it is a first step toward the second quantization (see, e.g., Bjorken and Drell, [28]) and underlies the demonstration of the Pauli principle. Let us show now that the fractal / nondifferentiable space-time approach of scale relativity allows one to easily recover the many-identical particle Schrödinger equation, and to understand the meaning of the non-separability between particles in the wave function.

The key to this problem clearly lies in the fact that, in the end, the wave function is nothing but another expression for the action, itself made complex as a consequence of nondifferentiability. We define a generalized, complex Lagrange function that depends on all positions and complex velocities of the particles:

$$\mathcal{L} = \mathcal{L}(x_1, x_2, ...x_n; \mathcal{V}_1, \mathcal{V}_2, ...\mathcal{V}_n; t), \tag{55}$$

from which the complex action keeps its form:

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}(x_1, x_2, ...x_n; \mathcal{V}_1, \mathcal{V}_2, ...\mathcal{V}_n; t)\, dt. \tag{56}$$

The wave function is still defined as:

$$\psi = e^{i\mathcal{S}/\hbar}, \tag{57}$$

with $\hbar = m\lambda$ still true, since all particles have the same mass. Then ψ is a function of the whole set of coordinates of the n particles. Reversing this equation yields:

$$\mathcal{S} = -i\hbar \ln \psi. \tag{58}$$

Up to now, we have essentially developed the Lagrangian formalism of our scale-covariant approach. Here it is more efficient to develop the Hamiltonian approach.

By considering the action integral as a function of the value of coordinates at the upper limit of integration, it is well-known that one gets two fundamental relations. The first is the expression for the momenta in terms of partial derivatives of the action. This relation has already been used in the one-particle case [equations (43) and (46)] and becomes:

$$\mathcal{P}_k = m\mathcal{V}_k = -im\lambda\nabla_k(\ln\psi). \tag{59}$$

where $k = 1$ to n is the particle index. For the particular solutions of the many-particle equation which are a *product* of solutions of the single-particle equation, they become a sum in terms of $\ln\psi$, so that the momentum of a given particle, being defined as the gradient of $\ln\psi$ relative to its own coordinates, depend only on them, and not on the coordinates of the other particles. However this is no longer true of the general solution, which is a linear combination of these particular ones.

The second relation is the expression for the Hamiltonian:

$$H = -\partial\mathcal{S}/\partial t. \tag{60}$$

Inserting into this equation the expression of the complex action (Eq.58), we finally get the general form of the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi, \tag{61}$$

provided the correspondence principle is also demonstrated for the energy / Hamiltonian. Let us achieve this demonstration by coming back to the Lagrangian approach, in the Newtonian case. The Lagrange function writes $L = \frac{1}{2} m \Sigma v_i^2$ and becomes in our approach the complex function $\mathcal{L} = \frac{1}{2} m \Sigma \mathcal{V}_i^2$. The sum is over three coordinates of n particles, and would be exactly the same if it were made over the $3n$ coordinates of a $3n$-dimensional space. This is true of the whole formalism developed in Sec. 4.1, so that we shall obtain the same Schrödinger equation as in equation (53), but with the 3-Laplacian Δ replaced by a $3n$-Laplacian, Δ_{3n}. Such an equation is then the same as equation (61), with the Hamiltonian H written as the sum of the one-body Hamiltonians, $H = \Sigma_i H(x_i, p_i)$. On the contrary, the $3n+1$ coordinates are mixed in our complex action, then also in the wave function $\psi = \psi(x_1, x_2, ... x_n; t)$.

4.3 Generalized Schrödinger equation

We shall now recall how one can relax some of the assumptions that lead to standard quantum mechanics, and obtain in this way generalized Schrödinger equations. This approach allows one to understand why the fractal / non fractal (i.e., in our interpretation quantum / classical) transition is so fast, and why the fractal dimension jumps directly from 1 to 2: the answer is, as can be seen in the following, that the generalized Schrödinger equation that one may construct for $1 < D < 2$ is degenerated and unphysical. The results described in this Section have been obtained in Refs. [12,14].

We assume that space is continuous and nondifferentiable, so that the geodesics of such a space are also expected to be continuous, nondifferentiable and in infinite number between any couple of points. This can be expressed by describing the position vector of a particle by a finite, continuous fractal 3-function $x(t, \delta t)$, explicitly dependent on the time resolution δt. Nondifferentiability also implies that the variation of the position vector between $t - dt$ and t and between t and $t + dt$ is described by two *a priori* different processes:

$$x(t + dt, dt) - x(t, dt) = v_+(x, t)\, dt + \zeta_+(t, dt)\, (dt/\tau_o)^{1/D}, \tag{62}$$

$$x(t, dt) - x(t - dt, dt) = v_-(x, t)\, dt + \zeta_-(t, dt)\, (dt/\tau_o)^{1/D}.$$

where D is the fractal dimension of the trajectory. The last terms in these equations can also be written in terms of fluctuations $d\xi_{\pm i}$, that are of zero mean , $< d\xi_{\pm i} >= 0$, and satisfy [see equation (28)]:

$$< d\xi_{\pm i} d\xi_{\pm j} >= \pm 2\mathcal{D} \delta_{ij} (dt^2)^{1/D}. \tag{63}$$

In this equation, $\mathcal{D}$ corresponds to a diffusion coefficient in the statistical interpretation of the theory in terms of a diffusion process. In the case considered up to now it was constant and related to the Compton length of the particle:

$$\mathcal{D} = \frac{\hbar}{2m} = c\lambda_c/2. \tag{64}$$

Two generalizations are particularly relevant: the case of a fractal dimension becoming different from 2 (as suggested by the development of scale relativity, see

Refs. [1,3,11] and below), and the case of a diffusion coefficient varying with position and time, $\mathcal{D} = \mathcal{D}(x,t)$, that must be considered in applications of this approach to chaotic dynamics (see Refs. [12,14] and in what follows). The general problem of tackling properly the case $D \neq 2$ lies outside the scope of the present contribution: such processes correspond to fractional Brownian motions, which are known to be non-Markovian, and persistent ($D < 2$) or antipersistent ($D > 2$). We shall consider only the case when the fractal dimension D is close to 2. Indeed, in this case its deviation from 2 can be approximated in terms of an explicit scale dependence on the time resolution, as first noticed by Mandelbrot and Van Ness [28]. Namely we write:

$$< d\xi_{\pm i} d\xi_{\pm j} >= \pm 2\mathcal{D}'(x,t)\delta_{ij} dt \times (\delta t/\tau)^{(2/D)-1}, \tag{65}$$

where τ is some characteristic time scale. Hence the effect of $D \neq 2$ can be dealt with in terms of a generalized, scale-dependent, "diffusion" coefficient:

$$\mathcal{D} = \mathcal{D}(x,t,\delta t) = \mathcal{D}'(x,t)(\delta t/\tau)^{(2/D)-1}. \tag{66}$$

We can now follow the lines of Ref [11] and Section 4.1. The complex time derivative operator (32) is found to be given by the same expression as in Eq. (36):

$$\frac{\acute{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\mathcal{D}(x,t,\delta t)\Delta. \tag{67}$$

but with $\mathcal{D}$ now a function given by Eq. (66).

The various steps of the subsequent demonstration Eqs. (38-44) are left unchanged. This is no longer the case in the following steps, since we must account for the diffusion coefficient being now a function of position. Rather than defining, as in [1,11], the probability amplitude by the relation $\psi = e^{i\mathcal{S}/2m\mathcal{D}}$, we define an average value of the diffusion coefficient, $< \mathcal{D} >$, that is a constant respectively to variables x and t, but may include an explicit scale-dependence in terms of time resolution δt,

$$\mathcal{D}(x,t) =< \mathcal{D} > +\delta\mathcal{D}(x,t), \tag{68}$$

and we introduce the complex function ψ from the relation,

$$\psi = e^{i\mathcal{S}/2m<\mathcal{D}>}. \tag{69}$$

Then ψ is related to the complex velocity :

$$\mathcal{V} = -2i < \mathcal{D} > \nabla(\ln\psi). \tag{70}$$

Our generalized complex Newton equation now takes the form

$$\nabla\mathcal{U} = 2im < \mathcal{D} > \frac{\acute{d}}{dt}(\nabla\ln\psi). \tag{71}$$

Equation (71) may be finally given the form of a generalized Schrödinger equation:

$$\nabla\left[\frac{\mathcal{U}}{2m < \mathcal{D} >} - \frac{1}{\psi}[\mathcal{D}\Delta\psi + i\frac{\partial\psi}{\partial t}] + \delta\mathcal{D}(\nabla\ln\psi)^2\right] = -\nabla(\delta\mathcal{D})\Delta\ln\psi. \tag{72}$$

The study of the general form of this equation is left open for future studies. Let us consider some special, simplified cases:

(i) $\mathcal{D} = \text{cst} = \hbar/2m$: in this case $\delta\mathcal{D} = 0$, and the last two terms disappear. The equation can be integrated and yields the Schrödinger equation [1,11].

(ii) $\mathcal{D} = \mathcal{D}(\delta t) = \mathcal{D}'(\delta t/\tau)^{(2/D)-1}$: this case corresponds to a diffusion coefficient which remains constant in terms of positions and time, but which includes the scale-depending effect of a fractal dimension different from 2. The last two terms of Eq.(72) also disappear in this case, so that we still obtain a Schrödinger-like equation:

$$\mathcal{D}^2(\delta t)\Delta\psi + i\mathcal{D}(\delta t)\frac{\partial\psi}{\partial t} = \frac{\mathcal{U}\psi}{2m} \tag{73}$$

The behavior of this equation is in agreement with the underlying stochastic process being no longer Markovian for $D \neq 2$. The "ultraviolet" ($\delta t << \tau$) and "infrared" ($\delta t >> \tau$) behaviors are reversed between the cases ($D < 2$) and ($D > 2$), but anyway they correspond to only two possible asymptotic behaviors, $\mathcal{D} \to \infty$ and $\mathcal{D} \to 0$:

(a) $\mathcal{D} \to \infty$ [UV $D > 2$; IR $D < 2$] : in this case Eq.(73) is reduced to $\Delta\psi = 0$, i.e., to the equation of a sourceless, stationarity probability amplitude. Whatever the field described by the potential $\mathcal{U}$, it is no longer feeled by the particle: this is nothing but the property of asymptotic freedom, which is already provided in quantum field theories by non abelian fields. This remark is particularly relevant for $D > 2$, since in this case this is the UV (i.e. small length-scale, high energy) behavior, and we shall recall hereafter that the principle of scale relativity leads one to introduce a scale-dependent generalized fractal dimension $D(r) > 2$ for virtual quantum trajectories considered at scales smaller than the Compton length of the electron.

(b) $\mathcal{D} \to 0$ [UV $D < 2$; IR $D > 2$] : Eq.(73) becomes completely degenerate ($\mathcal{U}\psi = 0$). Physics seems to be impossible under such a regime. This result is also in agreement with what is known about the quantum-classical transition. Indeed one finds, when trying to translate the behavior of typical quantum paths in terms of fractal properties [1], that their fractal dimension quickly jumps from $D = 2$ (quantum) to $D = 1$ (classical) when the resolution scale r becomes larger than the de Broglie scale. This fast transition actually prevents the domain $1 < D < 2$ to be achieved in nature (concerning the description of fractal space-time).

However this conclusion, that we reached in Ref. [12], must be now somewhat moderated. It applies, strictly, to D very different from 2 (and then indeed to the quantum / classical transition). But if one considers only a small perturbation to $D = 2$, then the asymptotic, degenerate behavior described above occurs only at very small scales, while one obtains at intermediate scales a new behavior of our generalized Schrödinger equation, namely, an explicit scale-dependence in terms of resolution. Such a behavior is not only physically meaningful, but even of the highest importance in the further development of the theory of scale relativity, since it may yield a clue to the nature of the electromagnetic field, of the electric charge and of gauge invariance (see [13] and herebelow Secs. 5.3 and 6.2).

(iii) $\nabla(\delta\mathcal{D}) = 0$ or $\nabla(\delta\mathcal{D}) << 1$: in this case, that corresponds either to a slowly varying diffusion coefficient in the domain considered, or, at the limit, to a diffusion coefficient depending on time but not on position, the right -hand side of Eq.(72) vanishes, so that it may still be integrated, yielding:

$$\mathcal{D}\Delta\psi + i\frac{\partial\psi}{\partial t} = \left[\frac{\mathcal{U}}{2m < \mathcal{D} >} + a + \delta\mathcal{D}(\nabla \ln\psi)^2\right]\psi, \tag{74}$$

where a is a constant of integration that can be made to vanish. Assuming that $\delta\mathcal{D}/\mathcal{D}$ remains $<< 1$, the effect of the term $\delta\mathcal{D}\,\psi(\nabla \ln\psi)^2$ which is in addition to the standard Schrödinger equation and the effect of $\mathcal{D}$ being a function of x and t can be treated perturbatively. One gets the equation

$$< \mathcal{D} >^2 \Delta\psi + i < \mathcal{D} > \frac{\partial\psi}{\partial t} = \left[\frac{\mathcal{U}(x,t)}{2m} - < \mathcal{D} > [\Delta \ln\psi]_0\, \delta\mathcal{D}(x,t)\right] \psi \tag{75}$$

Hence the effect of the new terms amounts to changing the form of the potential in the standard Schrödinger equation. Such a behavior could be of interest in the perspective of a future development of a field theory based on the concept of scale relativity and fractal space-time. Indeed, fluctuations in the fractal space-time geometry are expected to imply fluctuations $\delta\mathcal{D}(x,t)$ of the diffusion coefficient (in a way which remains to be described), which in turn will play the role of a potential in Eq. (75).

Let us conclude this Section by a brief comment about the complete equation (72). In the particular cases considered above, the statistical interpretation of the wave function ψ in terms of $\varrho = \psi\psi^\dagger$ giving the probability of presence of the particle remains correct, since the imaginary part of the Schrödinger equation is the equation of continuity (see Ref. [11] on this point and on the fact that we do not need to write the Fokker-Planck equations in our derivation of Schrödinger's equation). But this may no longer be the case for the general equation, since we took for our definition of ψ in Eq. (69) the simplest possible generalization, which may not be the adequate one. A more complete approach will be presented in a forthcoming work, a task which is revealed necessary in particular for applications of our method to chaotic dynamics (see below).

5 RELATIVISTIC QUANTUM MECHANICS

5.1 Free particle Klein-Gordon equation

Let us now come back to standard quantum mechanics, but in the motion-relativistic case. We shall recall here how one can get the free and electromagnetic Klein-Gordon equations, as already presented in Ref. [13].

Most elements of our approach as described in Sec. 3, remain correct, with the time differential element dt replaced by the proper time one, ds. Now not only space, but the full space-time continuum, is considered to be nondifferentiable, then fractal. The elementary displacement along a geodesics now writes (in the standard case $D = 2$):

$$dX^i_\pm = v^i_\pm ds + \lambda^{1/2} u^i_\pm ds^{1/2}. \tag{76}$$

We still define mean forward and backward derivatives, d_+/ds and d_-/ds :

$$\frac{d_\pm}{ds} y(s) = \lim_{\delta s \to 0\pm} < \frac{y(s+\delta s) - y(s)}{\delta s} > \tag{77}$$

which, once applied to x^i, yield *forward and backward mean 4-velocities*,

$$\frac{d_+}{ds} x^i(s) = v^i_+ \quad ; \quad \frac{d_-}{ds} x^i(s) = v^i_- . \tag{78}$$

The forward and backward derivatives of (78) can be combined in terms of a complex derivative operator [1,13]

$$\frac{\acute{d}}{ds} = \frac{(d_+ + d_-) - i\,(d_+ - d_-)}{2ds}, \tag{79}$$

which, when applied to the position vector, yields a complex 4-velocity

$$\mathcal{V}^i = \frac{d}{ds}x^i = V^i - i\,U^i = \frac{v_+^i + v_-^i}{2} - i\,\frac{v_+^i - v_-^i}{2}. \tag{80}$$

We must now jump to the stochastic interpretation of the theory, due to the infinity of geodesics of the fractal space-time. This forces us to consider the question of the definition of a Lorentz-covariant diffusion in space-time. This problem has been addressed by several authors in the framework of a relativistic generalization of Nelson's stochastic quantum mechanics. Forward and backward fluctuations, $d\xi_\pm^i(s)$, are defined, which are Gaussian with mean zero, mutually independent and such that

$$< d\xi_\pm^i\, d\xi_\pm^j >= \mp\lambda\eta^{ij}\,ds. \tag{81}$$

Once again, the identification with a diffusion process allows one to relate λ to the diffusion coefficient, $2\mathcal{D} = \lambda c$, with $\mathcal{D} = \hbar/2m$. But the difficulty comes from the fact that such a diffusion makes sense only in R^4, i.e. the "metric" η^{ij} should be positive definite, if one wants to interpret the continuity equation satisfied by the probability density (see hereafter) as a Kolmogorov equation. Several proposals have been made to solve this problem.

Dohrn and Guerra [30] introduce the above "Brownian metric" and a kinetic metric g_{ij}, and obtain a compatibility condition between them which reads $g_{ij}\eta^{ik}\eta^{jl} = g^{kl}$. An equivalent method was developed by Zastawniak [31], who introduces, in addition to the covariant forward and backward drifts v_+^i and v_-^i (please note that our notations are different from his) new forward and backward drifts b_+^i and b_-^i. In terms of these, the Fokker-Planck equations (that one can derive in our approach from the Klein-Gordon equation) become Kolmogorov equations for a standard Markov-Wiener diffusion in $\mathbb{R}^4$. Serva [32] gives up Markov processes and considers a covariant process which belongs to a larger class, known as "Bernstein processes".

All these proposals are equivalent, and amount to transforming a Laplacian operator in $\mathbb{R}^4$ into a Dalembertian. Namely, the two forward and backward differentials of a function $f(x, s)$ write (we assume a Minkowskian metric for classical space-time):

$$d_\pm f/ds = (\partial/\partial s + v_\pm^i.\partial_i \mp \frac{1}{2}\lambda\partial^i\partial_i)f. \tag{82}$$

In what follows, we shall only consider s-stationary functions, i.e., that are not explicitly dependent on the proper time s . In this case our time derivative operator reduces to:

$$\frac{\acute{d}}{ds} = \left(\mathcal{V}^k + \frac{1}{2}i\,\lambda\,\partial^k\right)\partial_k. \tag{83}$$

(Note the correction of sign with respect to Ref. [13]. The sign + of the Dalambertian comes from the choice of a metric $(+, -, -, -)$ for the classical space-time. See Pissondes [33] for more detail). We shall now generalize to the relativistic case our demonstration (Refs. [1,11-15] and Sections 3 and 4) that the passage from classical (differentiable) mechanics to quantum mechanics can be implemented by a unique prescription: replace the standard time derivative d/ds by the new complex operator $\acute{d}/ds$ that plays the role of a quantum-covariant derivative.

Let us first note that Eq. (83) can itself be derived from the introduction of a partial quantum-covariant derivative $\acute{d}_k = \acute{d}/dx^k$:

$$\acute{d}_k = \partial_k + \frac{1}{2} i\, \lambda\, \frac{\mathcal{V}_k}{\mathcal{V}^2}\, \partial^j \partial_j, \tag{84}$$

where $\mathcal{V}^2 = \mathcal{V}_k \mathcal{V}^k$. It is easy to check that $\acute{d}/ds = \mathcal{V}^k \acute{d}_k$. Let us assume that any mechanical system can be characterized by a stochastic (complex) action \$. The same reasoning as in classical mechanics leads us to write $d\$^2 = -m^2c^2\mathcal{V}_k\mathcal{V}^k ds^2$. The least-action principle applied on this action yields the equations of motion of a free particle, $\acute{d}\mathcal{V}_k/ds = 0$. We can also write the variation of the action as a functional of coordinates. We obtain the usual result (but here generalized to complex quantities):

$$\delta\$ = -mc\mathcal{V}_k \delta x^k \;\Rightarrow\; \mathcal{P}_k = mc\mathcal{V}_k = -\partial_k \$ \tag{85}$$

where $\mathcal{P}_k$ is now a complex 4-momentum. As in the nonrelativistic case, the wave function is introduced as being nothing but a reexpression of the action:

$$\psi = e^{i\$/mc\lambda} \;\Rightarrow\; \mathcal{V}_k = i\lambda\partial_k(\ln\psi), \tag{86}$$

so that the equations of motion ($\acute{d}\mathcal{V}_k/ds = 0$) become:

$$\acute{d}\mathcal{V}_j/ds = i\lambda\left(\mathcal{V}^k\partial_k + \frac{1}{2}i\lambda\partial^k\partial_k\right)\partial_j(\ln\psi) = 0. \tag{87}$$

Arrived at this stage, it is easy to show that this equation amounts to the Klein-Gordon one. Indeed, if we replace the time variable t by it in this equation, the Dalembertian is replaced by a 4-Laplacian (with a change of sign), and we are brought back to the non-relativistic problem which has already been treated in detail in Refs. [1,11]. So Eq. (87) can finally be put under the form of a vanishing four-gradient which is integrated in terms of the Klein-Gordon equation for a free particle:

$$\partial_j\left[\frac{\lambda^2\partial^k\partial_k\psi}{\psi}\right] = 0 \;\Rightarrow\; \lambda^2\partial^k\partial_k\psi = \psi. \tag{88}$$

We recall that $\lambda = \hbar/mc$ is the Compton length of the particle. The integration constant is 1 in order to ensure the identification of $\varrho = \psi\psi^\dagger$ with a probability density for the particle.

Before going on with the introduction of fields from the scale-relativistic approach, we want to stress the physical meaning of the above result. While Eq. (88) is the equation of a free quantum particle, the equation of motion from which we started takes exactly the form of the classical equation of a free particle, $d^2x^k/ds^2 = 0$. Quantum effects and quantum behavior appear here through the implementation of scale covariance, as manifestation of the nondifferentiable and fractal nature of the micro-space-time.

5.2 Nature of the electromagnetic field

The theory of scale relativity also allows us to get new insights about the nature of the electromagnetic field, of the electric charge, and the physical meaning of gauge invariance [12]. Consider an electron (or any other particle). In scale relativity, we identify the particle with its potential fractal trajectories, described as the geodesics of a nondifferentiable space-time. These trajectories are characterized by internal (fractal) structures. Now consider anyone of these structures, lying at some (relative) resolution ε (such that $\varepsilon < \lambda$) for a given position of the particle. In a displacement of the particle, the relativity of scales implies that the resolution at which this given structure appears in the new position will *a priori* be different from the initial one. In other words, we expect the occurrence of *dilatations of resolutions induced by translations*, which read:

$$e\frac{\delta\varepsilon}{\varepsilon} = -A_\mu \delta x^\mu. \tag{89}$$

Under this form, the dimensionality of A_μ is CL^{-1}, where C is the electric charge unit (note the correction to the notations of Ref. [12]). This behaviour can be expressed in terms of a scale-covariant derivative:

$$e\, D_\mu \ln(\lambda/\varepsilon) = e\, \partial_\mu \ln(\lambda/\varepsilon) + A_\mu. \tag{90}$$

However, if one wants such a "field" to be physical, it must be defined whatever the initial scale from which we started. Starting from another scale $\varepsilon' = \varrho\varepsilon$ (we consider Galilean scale-relativity for the moment), we get

$$e\,\frac{\delta\varepsilon'}{\varepsilon'} = -A'_\mu\, \delta x^\mu, \tag{91}$$

so that we obtain:

$$A'_\mu = A_\mu + e\, \partial_\mu \ln \varrho, \tag{92}$$

which depends on the relative "state of scale" , $\mathbb{V} = \ln \varrho = \ln(\varepsilon/\varepsilon')$. However, if one now considers translation along two different coordinates (or, in an equivalent way, displacement on a closed loop), one may write a commutator relation:

$$e\,(\partial_\mu D_\nu - \partial_\nu D_\mu) \ln(\lambda/\varepsilon) = (\partial_\mu A_\nu - \partial_\nu A_\mu). \tag{93}$$

This relation defines a tensor field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ which, contrarily to A_μ, is independent of the initial scale. One recognizes in $F_{\mu\nu}$ an electromagnetic field, in A_μ an electromagnetic potential and in (92) the property of gauge invariance which, in accordance with Weyl's initial ideas [34], recovers its initial status of scale invariance. However, equation (92) represents a progress compared with these early attempts and with the status of gauge invariance in today's physics. Indeed the gauge function, which has, up to now, been considered as arbitrary and devoid of physical meaning, is now identified with the logarithm of internal resolutions. In Weyl's theory [34], and in its formulation by Dirac [35], the metric element ds (and consequently the length of any vector) is no longer invariant and can vary from place to place in terms of some (arbitrary) scale factor. Such a theory was excluded by experiment, namely by the existence of universal and unvarying lengths such as the electron Compton length. In scale relativity, we are naturally led to introduce two "proper times" , the classical one ds which remains invariant, and the fractal one $d\mathcal{S}$, which is scale-divergent and can then vary from place to place. In Galilean scale-relativity, the fractal dimension of geodesics is $D = 2$, so that the scale-dependence of $d\mathcal{S}$ writes $d\mathcal{S} = d\sigma(\lambda/\varepsilon)$ (see Ref. [13]). Therefore we have $\delta(d\mathcal{S})/d\mathcal{S} = -\delta\varepsilon/\varepsilon \propto A_\mu \delta x^\mu$, and we recover the basic relation of the Weyl-Dirac theory, in the asymptotic high energy domain ($\varepsilon < \lambda$).

As we shall see in what follows, the passage to quantum theory and to Lorentzian scale relativity will allow us to derive a general mass-charge relation (Section 6.3) and to make new theoretical predictions. But first recall here how one can recover the Klein-Gordon equation with electromagnetic field. We introduce a generalized action which now *a priori* depends on motion and on scale variables, $\$ = \$(x, \mathcal{V}, \mathbb{V})$:

$$d\$ = \frac{\partial \$}{\partial x^\mu} dx^\mu + \frac{\partial \$}{\partial \mathbb{V}} d\mathbb{V}. \tag{94}$$

The first term is the standard, free-particle term, while the second term is a new scale contribution. In the case that we consider here where scale laws and motion laws are coupled together, $\mathbb{V}$ depends on coordinates, so that $d\$ = (\partial\$/\partial\mathbb{V}) D_\mu \mathbb{V} dx^\mu = (\partial\$/\partial x^\mu) dx^\mu + (\partial\$/\partial\mathbb{V})(1/e) A_\mu dx^\mu$. Now we shall see in the next section (equation 106) that $\partial\$/\partial\mathbb{V} = -e^2/c$, so that we can finally write the action differential under the form:

$$d\$ = -i\hbar d \ln\psi = -mc\mathcal{V}_\mu dx^\mu - \frac{e}{c} A_\mu dx^\mu. \tag{95}$$

Equation (95) allows one to define a new generalized complex four-momentum,

$$\tilde{\mathcal{P}}^\mu = \mathcal{P}^\mu + \Delta P^\mu = mc\mathcal{V}^\mu + (e/c)A^\mu. \tag{96}$$

This leads us to introduce a 'covariant' velocity:

$$\tilde{\mathcal{V}}_\mu = i\lambda\partial_\mu(\ln\psi) - \frac{e}{mc^2} A_\mu, \tag{97}$$

and we recognize the well-known QED-covariant derivative, in agreement with (90),

$$-i\hbar D_\mu = -i\hbar\partial_\mu + (e/c)A_\mu, \tag{98}$$

since we can write (97) as $mc\tilde{\mathcal{V}}_\mu\psi = [i\hbar\partial_\mu - (e/c)A_\mu]\psi$. The scale-covariant free particle equation

$$d^2x^\nu/ds^2 = 0 \rightarrow \,\not{d}^{\,2}x^\nu/ds^2 = \left(\mathcal{V}_\mu + i\frac{\lambda}{2}\partial_\mu\right)\partial^\mu\mathcal{V}^\nu = \left(\mathcal{V}_\mu + i\frac{\lambda}{2}\partial_\mu\right)\partial^\nu\mathcal{V}^\mu = 0 \tag{99}$$

can now be made also QED-covariant. Indeed the substitution $\mathcal{V}_\mu \rightarrow \tilde{\mathcal{V}}_\mu$ in the last form of (99) yields:

$$\left(\tilde{\mathcal{V}}_\mu + i\frac{\lambda}{2}\partial_\mu\right)\partial^\nu\tilde{\mathcal{V}}^\mu = 0. \tag{100}$$

Once integrated, equation (100) takes the form of the Klein-Gordon equation for a particle in an electromagnetic field,

$$[i\hbar\partial_\mu - (e/c)A_\mu][i\hbar\partial^\mu - (e/c)A^\mu\psi = m^2c^2\psi. \tag{101}$$

More detail about this demonstration will be given in Ref. [33]. The case of the Dirac equation will be considered in forthcoming works. See Refs [7,36,37] for early attempts to comprehend it in terms of fractal and stochastic models.

5.3 Nature of the electric charge

In a gauge transformation $A_\mu = A_{0\mu} - \partial_\mu \chi$ the wave function of an electron of charge e becomes:

$$\psi = \psi_0 \, e^{ie\chi}. \tag{102}$$

In this expression, the essential role played by the so-called "arbitrary" gauge function is clear. It is the variable conjugate to the electric charge, in the same way as position, time and angle are conjugate variables of momentum, energy and angular momentum in the expressions for the action and/or the quantum phase of a free particle, $\theta = (px - Et + \sigma\varphi)$. Our knowledge of what are energy, momentum and angular momentum comes from our understanding of the nature of space, time, angles and their symmetry, via Noether's theorem. Conversely, the fact that we still do not really know what is an electric charge despite all the development of gauge theories, comes, in our point of view, from the fact that the gauge function χ is considered devoid of physical meaning.

We have reinterpreted in the previous section the gauge transformation as a scale transformation of resolution, $\varepsilon_0 \to \varepsilon, \mathbb{V} = \ln \varrho = \ln(\varepsilon_0/\varepsilon)$. In such an interpretation, the specific property that characterizes a charged particle is the explicit scale-dependence on resolution of its action, then of its wave function. The net result is that the electron wave function writes

$$\psi = \psi_0 \exp\left\{ i \frac{e^2}{\hbar c} \mathbb{V} \right\}. \tag{103}$$

Since, by definition (in the system of units where the permittivity of vacuum is 1),

$$e^2 = 4\pi\alpha\hbar c, \tag{104}$$

equation (103) becomes

$$\psi = \psi_0 \, e^{i 4\pi\alpha \mathbb{V}}. \tag{105}$$

This result allows us to suggest a solution to the problem of the nature of the electric charge (and also yields new mass-charge relations, see hereafter Section 6.3). Indeed, considering now the wave function of the electron as an explicitly resolution-dependent function, we can write the scale differential equation of which ψ is solution as:

$$-i\hbar \frac{\partial \psi}{\partial \left(\frac{e}{c}\mathbb{V}\right)} = e\psi. \tag{106}$$

We recognize in $\tilde{D} = -i\hbar\partial/\partial((e/c)\ln\varrho)$ a dilatation operator $\tilde{D}$ similar to that introduced in Section 2 [equation (3)]. Equation (106) can then be read as an eigenvalue equation issued from a new application of the correspondence principle,

$$\tilde{D}\psi = e\psi. \tag{107}$$

In such a framework, the electric charge is understood as the conservative quantity that comes from the new scale symmetry, namely, the uniformity of the resolution variable $\ln \varepsilon$.

Let us finally remark, as already done in Ref. [13], that only global dilations of resolutions $\varepsilon'_\mu = \varrho\varepsilon_\mu$ have been considered here as a simplifying first step. But the theory allows one to define four different and independent dilations along the four space-time resolutions. It is then clear that the electromagnetic field is expected to be embedded into a larger field and the electric charge to be one element of a more complicated, "vectorial" charge. We shall consider in forthcoming works the possibility to recover in this way the electroweak theory or a generalization of it [19].

6 IMPLICATION OF SPECIAL SCALE RELATIVITY

Let us now jump again to the special scale-relativistic version of the theory, and examine its possible theoretical and experimental consequences. As recalled in Section 2, we have suggested that today's physical theories are only large scale approximations of a more profound theory in which the laws of dilation take a Lorentzian structure. Scale-relativistic "corrections" remain small at "large" scale (i.e., around the Compton scale of the electron), and increase when going to smaller length scales (i.e., large energies), in the same way as motion-relativistic correction increase when going to large speeds. The results described in this section have been presented in Refs [1,11-13].

6.1 Coupling constants of fundamental interactions

It is clear that the new status of the Planck length-scale as a lowest unpassable scale must be universal. In particular, it must apply also to the de Broglie and Compton scales themselves, while in their standard definition they may reach the zero length. The de Broglie and Heisenberg relations then need to be generalized. We have presented in Refs. [1,3] the construction of a "scale-relativistic mechanics" that allows such a generalization. But there is a very simple way to recover the result that was obtained. We have shown in [2] that the generalization to any fractal dimension $D = 1+\delta$ of the de Broglie and Heisenberg relations wrote $p/p_0 = (\lambda_0/\lambda)^\delta$, where p_0 is the average momentum of the particle, and $\sigma_p/p_0 = (\lambda_0/\sigma_x)^\delta$. Scale covariance suggests that these results are conserved, but with δ now depending on scale as given by Eq.(83), which is precisely the result of Ref. [3]. As a consequence the mass-energy scale and length scale are no longer inverse, but related by the scale-relativistic generalized Compton formula (already given in Section 2):

$$\ln \frac{m}{m_0} = \frac{\ln(\lambda_0/\lambda)}{\sqrt{1 - \frac{\ln^2(\lambda_0/\lambda)}{\ln^2(\lambda_0/\mathbf{\Lambda})}}}, \tag{108}$$

i.e., $m/m_0 = (\lambda_0/\lambda)^{\delta(\lambda)}$, with $\delta(\lambda_0) = 1$.

Concerning coupling constants, the fact that the lowest order terms of their β -function are quadratic [i.e., their renormalization group equation reads $d\alpha/d\mathbb{V} = \beta_0\alpha^2 + O(\alpha^3)$] implies that their variation with scale is unaffected to this order by scale-relativistic corrections [1], *provided it is written in terms of length scale.* The passage to mass-energy scale is now performed by using (108).

Let us briefly recall some of the results which have been obtained in this new framework:

•*Scale of Grand Unification:* Because of the new relation between length-scale and mass-scale, the theory yields a new fundamental scale, given by the length-scale corresponding to the Planck energy. This scale plays for scale laws the same role as played by the Compton scale for motion laws. Indeed, the Compton scale is $\hbar/mc$, while the velocity for which $p = mc$ is $c/\sqrt{2}$. Similarly, the new scale is given to lowest order by the relation

$$\ln(\lambda_Z/\lambda_p) = \mathbb{C}_Z/\sqrt{2}, \tag{109}$$

[where $\mathbb{C}_Z \approx \ln(m_P/m_Z)$]: it is $\approx 10^{-12}$ times smaller than the Z length-scale. In other words, this is but the GUT scale ($\approx 10^{14}$ GeV in the standard non scale-relativistic theory) in the minimal standard model with "great desert hypothesis", i.e., no new elementary particle mass scale beyond the top quark [1,3].

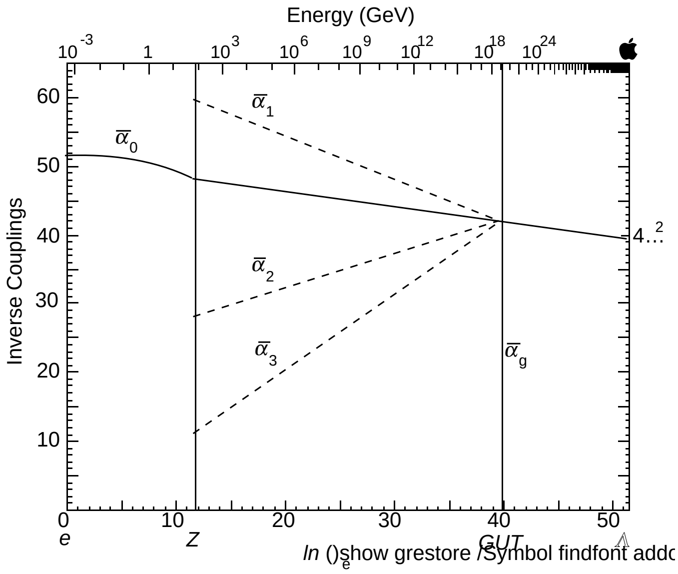

Figure 7: Variation of the QCD inverse coupling between infinite energy scale (i.e., Planck length-scale in special scale-relativity) and electron scale (see text). The fine structure constant is only the 3/8 of α_0, because the weak bosons have acquired mass in the Higgs mechanism. Only an approximation of the α_1, α_2, and α_3 couplings are needed in this calculation of α_0. (See Fig. 8 for a more precise description of their variation).

• *Unification of ChromoElectroWeak and Gravitational fields*: As a consequence, the four fundamental couplings, U(1), SU(2), SU(3) and gravitational converge in the new framework towards about the same scale, which now corresponds to the Planck mass scale. The GUT energy now being of the order of the Planck one ($\approx 10^{19}$ GeV), the predicted lifetime of the proton ($\propto m_{GUT}^4/m_p^5 >> 10^{38}$ yrs) becomes compatible with experimental results ($> 5.5 \times 10^{32}$ yrs) [1,3].

•*Fine structure constant*: The problem of the divergence of charges (coupling constants) and self-energy is solved in the new theory. They have finite non-zero values at infinite energy in the new framework, while in the standard model they were either infinite (Abelian U(1) group) either null (asymptotic freedom of non-Abelian groups). Such a behavior of the standard theory prevented one from relating the "bare" (infinite energy) values of charges to their low energy values, while this is now possible in the scale-relativistic standard model. We find indeed that the formal QED inverse coupling $\bar{\alpha}_0 = \frac{3}{8}\bar{\alpha}_2 + \frac{5}{8}\bar{\alpha}_1 = \frac{3}{8}\bar{\alpha}$ (where $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are respectively the U(1) and SU(2) inverse couplings), when "runned" from the electron scale down to the Planck length-scale by using its renormalization group equation, converges towards the value $39.46 \pm 0.05 = 4 \times (3.1411 \pm 0.0019)^2 \approx 4\pi^2$ at infinite energy (see Figs 7 and 8) [1,3,11].

Let us give a simple new argument supporting the idea that $1/4\pi^2$ might indeed correspond to an optimization process for a coupling constant. Assume that a Coulomb-like force between two "charges" is transported by some intermediate particles. This force is given by $F =< \delta p/\delta t >$, the average variation of momentum over the time interval δt. Each actual exchange of momentum is optimized in terms of the minimum of the Heisenberg inequality for intervals, that can be derived from the general method devised by Finkel [38]. Namely, defining $\delta r = |(r- < r > |$ ($= r$ for $< r >= 0$) and $\delta p = |(p- < p > |$, he finds that the Heisenberg relation for these variables reads $\delta p \times \delta r \geq \hbar/\pi$. For interactions propagating at the velocity of light we have $\delta t = r/c$. Now, not all intermediate particles emitted by one of the interacting particle are received by the other, but only a fraction $1/4\pi$. We finally get a force $F = (1/4\pi)\hbar c/\pi r^2 = \alpha \hbar c/r^2$, leading to a coupling constant $\alpha = 1/4\pi^2$.

The observed, low energy fine structure constant ($\approx 1/137.036$) would then result from such a bare coupling (1/39.478), decreased by a factor 3/8 due to the acquiring of a mass by 3 out of 4 gauge bosons ($\rightarrow 1/105.276$), then by a factor $\approx 75\%$ due to the polarization of the vacuum by particle-antiparticle pairs between the electroweak and electron scale (see the detailed calculation below).

Conversely, the conjecture that the corresponding "bare charge" $\alpha^{1/2}$ is given by $1/2\pi$ allowed us to obtain a theoretical estimate of the low energy fine structure constant to better than 1 of its measured value [11,19], and to predict that the number of Higgs doublets, which contributes to $2.11 N_H$ in the final value of $\bar{\alpha}, is N_H = 1$.

Let us sum up the calculations that led us to these results. The running of the inverse fine structure constant from its infinite energy value to its low energy (electron scale) value reads [11,19]:

$$\bar{\alpha}(\lambda_e) = \bar{\alpha}(\mathbf{\Lambda}) + \triangle\bar{\alpha}^{(1)}_{\mathbf{\Lambda}Z} + \triangle\bar{\alpha}^{(2)}_{\mathbf{\Lambda}Z} + \triangle\bar{\alpha}^{L}_{Ze} + \triangle\bar{\alpha}^{h}_{Ze} + \triangle\bar{\alpha}^{\mathrm{Sc-rel}} \tag{110}$$

where $\bar{\alpha}(\mathbf{\Lambda}) = \bar{\alpha}(E = \infty) = 32\pi^2/3$; $\Delta\bar{\alpha}^{(1)}_{\mathbf{\Lambda}Z}$ is the first order variation of the inverse coupling between the Planck length-scale (i.e., infinite energy in the new framework) and the Z boson length-scale, as given by the solution to its renormalization group equation [1],

$$\Delta\bar{\alpha}^{(1)}_{\mathbf{\Lambda}Z} = \frac{10 + N_H}{6\pi} \ln\frac{\lambda_Z}{\mathbf{\Lambda}} = \frac{10 + N_H}{6\pi} \mathbb{C}_Z = 23.01 + 2.11(N_H - 1); \tag{111}$$

$\Delta\bar{\alpha}^{(2)}_{\mathbf{\Lambda}Z}$ is its second order variation, which now depends on the three fundamental couplings α_1, α_2 and α_3 (which may themselves be estimated thanks to their

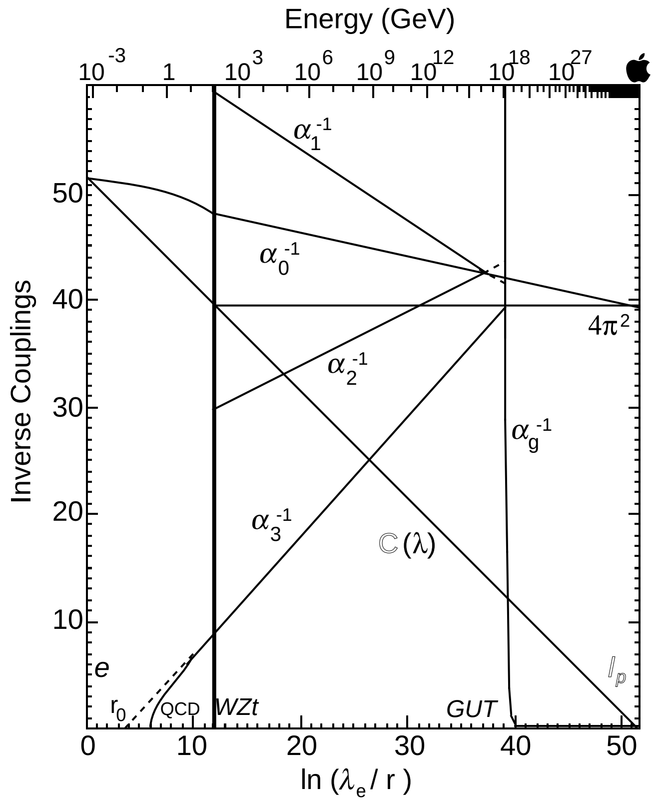

Figure 8: Variation with scale of the inverse couplings of the fundamental interactions U(1), SU(2) and SU(3) in the scale-relativistic minimal standard model. This scale-relativistic diagram (mass-scale versus inverse coupling constants) shows well-defined structures and symmetries that are accounted for by our new mass / charge relations (see text).

renormalization group equations) [11,19]:

$$\Delta\bar{\alpha}^{(2)}_{\mathbf{\Lambda}Z} = -\frac{104+9N_H}{6\pi(40+N_H)}\ln\left\{1-\frac{40+N_H}{20\pi}\alpha_1(\lambda_Z)\ln\frac{\lambda_Z}{\mathbf{\Lambda}}\right\}$$

$$+\frac{20+11N_H}{2\pi(20-N_H)}\ln\left\{1+\frac{20-N_H}{12\pi}\alpha_2(\lambda_Z)\ln\frac{\lambda_Z}{\mathbf{\Lambda}}\right\}+\frac{20}{21\pi}\ln\left\{1+\frac{7}{2\pi}\alpha_3(\lambda_Z)\ln\frac{\lambda_Z}{\mathbf{\Lambda}}\right\}$$
$$= 0.73 \pm 0.03 \quad (112)$$

$\Delta\bar{\alpha}^{L}_{Ze}$ is the leptonic contribution to its variation between electron and Z scales [11,19]:

$$\Delta\bar{\alpha}^{L}_{Ze} = \frac{2}{3\pi}\left\{\ln\left(\frac{m_Z}{m_e}\right)+\ln\left(\frac{m_Z}{m_\mu}\right)+\ln\left(\frac{m_Z}{m_\tau}\right)-\frac{5}{2}\right\} = 4.30 \pm 0.05; \quad (113)$$

$\Delta\bar{\alpha}^{h}_{Ze}$ is the hadronic contribution to its variation between electron and Z scales, which can be precisely inferred from the experimental values of the ratio R of the cross sections $\sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ [39]

$$\Delta\bar{\alpha}^h_{Ze} = 3.94 \pm 0.12; \tag{114}$$

and $\Delta\bar{\alpha}^{\mathrm{Sc-rel}} = -0.18 \pm 0.01$ is the scale-relativistic correction which comes from the fact that the length-scales and mass-scales of elementary particles are no longer directly inverse in the new framework. Combining all these contributions we have obtained [11,19]

$$\bar{\alpha}(\lambda_e) = 137.08 + 2.11(N_H - 1) \pm 0.13, \tag{115}$$

in very good agreement with the experimental value 137.036, provided $N_H = 1$ as announced above.

•*QCD coupling*: The SU(3) inverse coupling may be shown to cross the gravitational inverse coupling at also the same value $\bar{\alpha}_3 = 4\pi^2$ at the Planck mass-scale (more precisely for a mass scale $m_P/2\pi$). This allows one to get a theoretical estimate for the value of the QCD coupling at Z scale. Indeed its renormalization group equation yields a variation of $\bar{\alpha}_3$ with scale given to second order by:

$$\bar{\alpha}_3(r) = \bar{\alpha}_3(\lambda_Z) + \frac{7}{2\pi}\ln\frac{\lambda_Z}{r} + \frac{11}{4\pi(40+N_H)}\ln\left\{1 - \frac{40+N_H}{20\pi}\alpha_1(\lambda_Z)\ln\frac{\lambda_Z}{r}\right\}$$
$$-\frac{27}{4\pi(20-N_H)}\ln\left\{1 + \frac{20-N_H}{12\pi}\alpha_2(\lambda_Z)\ln\frac{\lambda_Z}{r}\right\} + \frac{13}{14\pi}\ln\left\{1 + \frac{7}{2\pi}\alpha_3(\lambda_Z)\ln\frac{\lambda_Z}{r}\right\}. \tag{116}$$

This leads to the prediction: $\alpha_3(m_Z) = 0.1155 \pm 0.0002$ [11,19], that compares well with the present experimental value, $\alpha_3(m_Z) = 0.112 \pm 0.003$.

•*Electroweak scale and solution to the hierarchy problem*: We shall see in what follows that the electroweak / Planck scale ratio is also determined by the same number, i.e., by the bare inverse coupling $\bar{\alpha}_0(\infty) \approx 4\pi^2$ [1,11,13]. We shall vindicate this claim in what follows by the demonstration of a mass-charge relation of the form $\alpha\mathbb{C} = k/2$, with k integer, coming from our new interpretation of gauge invariance [13]. This solves the hierarchy problem: the Planck scale energy is $\approx 10^{19}$ GeV, while the W and Z boson energy is $\approx 10^2$ GeV. This ratio 10^{17} is understood in a simple way in scale relativity as identified with:

$$e^{4\pi^2} = 1.397 \times 10^{17}. \tag{117}$$

Even if we disregard our $4\pi^2$ conjecture, we can compute $\bar{\alpha}_0(\infty)$ from its low energy value and from its scale variation as derived from its renormalization group equation [see equations (110-117) above]. We found $\bar{\alpha}_0(\infty) = 39.46 \pm 0.05$, so that we get a scale ratio:

$$e^{\bar{\alpha}_0(\infty)} = (1.37 \pm 0.07)10^{17}. \tag{118}$$

More precisely, the relation $\ln(m_P/m) = \bar{\alpha}_0(\infty) = 4\pi^2$ yields a mass $m_{WZ} = 87.393$ GeV, closely connected to the W and Z boson masses (currently $m_Z = 91.182$ GeV, $m_W = 80.0$ GeV). Moreover, the new fundamental scale λ_V given by the mass-charge relation $\mathbb{C}_V = \bar{\alpha}_0(\infty) = 4\pi^2$ (see below) corresponds to a mass scale

$$m_V = 123.23(1)\mathrm{GeV}. \tag{119}$$

Such a scale seems to be very directly connected with the vacuum expectation value of the Higgs field (currently $f \approx 174$ GeV $\approx 123\sqrt{2}$ GeV) [1,11].

6.2 Fermion Mass spectrum.

One of the most encouraging results of the theory of scale relativity in the microphysical domain is its ability to suggest a possible mechanism of generation of the mass spectrum of elementary particles. As we shall see, this mechanism is able to generate the mass and charge spectrum of charged elementary fermions in terms of the electron mass and of the electron charge, that are taken as free parameters at this level of the analysis. But note that these two last quantities are themselves related by a mass / charge relation (see next section), and that the low energy electric charge can be deduced from its infinite energy ("bare") value (Section 6.1).

A first description of this generation mechanism has been given in [11]. It is based on the observation that scale-relativistic "corrections" [Eq. (122) below] closely follow the variation of the electric charge due to vacuum polarization by particle-antiparticle pairs of elementary fermions (see Fig. 9). While this empirical law clearly remains true and remarkable, though it actually still depend on a free parameter (k in Figs. 9 and 10), our first attempt at understanding it theoretically in terms of variation of mass with scale [11, 13] was in error, as specified in Ref. [12]. In the present contribution, we shall only briefly recall the general principle of the method, then we shall suggest a possible road toward an understanding of this relation in terms of internal fractal structures of the electron geodesics.

As a first step toward the construction of our generation mechanism, let us show that there is a possible relation between the scale variation of charges due to radiative corrections and perturbation of the fractal dimension with respect to the critical value $D = 2$. The fractal fluctuation [Eqs. (30) and (81)], that contributes to the additional second order terms that must be introduced in differential equations (Eq.34), writes in the case $D = 1 + \delta \neq 2$:

$$\frac{< d\xi^2 >}{ds} = \lambda \left(\frac{ds}{\lambda}\right)^{(1-\delta)/(1+\delta)} . \tag{120}$$

As recalled in Section 4.3, the effect of the fractal dimension difference can be expressed in terms of an explicit scale dependence of the transition scale λ (that is equivalent to the 'diffusion coefficient' in the diffusion interpretation of the theory). Since $\lambda = \hbar/mc$, this scale dependence can be attributed to an effective Planck constant $\tilde{\hbar}$:

$$\tilde{\hbar} = \hbar \left(\frac{ds}{\lambda}\right)^{(1-\delta)/(1+\delta)} \approx \hbar \left\{1 + \frac{\delta - 1}{\delta + 1} \ln\left(\frac{\lambda}{ds}\right)\right\} . \tag{121}$$

Since we consider here only small perturbations to $D = 2$, we have $\ln(\lambda/ds) \approx 2\ln(\lambda/r) = 2\mathbb{V}$, where $r = \delta X (\approx \delta\xi \approx \hbar/p$ in the asymptotic domain) and our effective $\tilde{\hbar}$ reads to lowest order:

$$\tilde{\hbar} \approx \hbar\{1 + (\delta - 1)\mathbb{V}\}. \tag{122}$$

Once written in terms of $\tilde{\hbar}$, the Schrödinger equation and its solutions keep their form: this means that the transformation $\hbar \to \tilde{\hbar}$ is once again a form of covariance.

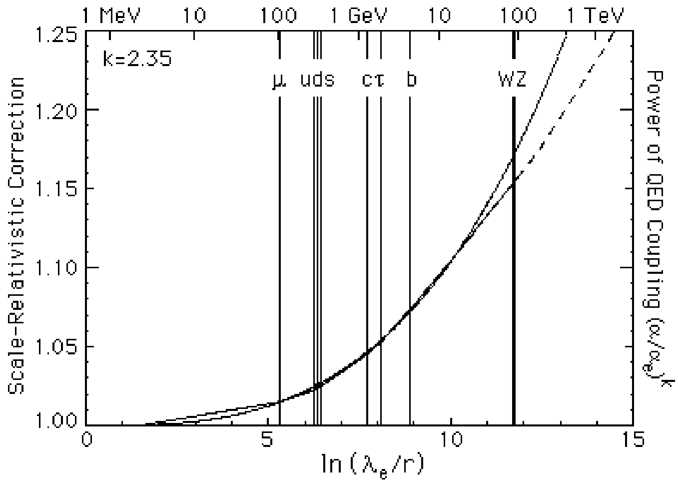

Figure 9: Comparison between the special scale-relativistic correction (square-root of Eq. 129) and the variation of the fine structure constant (to the power k) due to the pairs of elementary charged fermions.

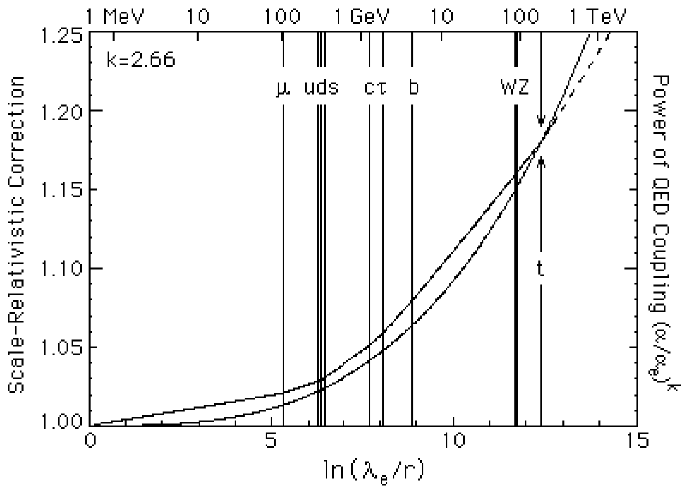

Figure 10: Same as Fig. 9, with a different choice for the power k. In all cases, the last elementary fermion in our theory (observed to be actually the top quark) is predicted to have a mass just larger than the W/Z mass.

Now starting from the electron solution (Eq.103), $\hbar$ will be replaced by $\tilde{h}$ in its expression, and the new wave function will read to lowest order:

$$\psi = \psi_0 exp \left\{ i \frac{e^2}{\hbar c} \mathbb{V} [1 - (\delta - 1) \mathbb{V}] \right\} . \tag{123}$$

This means that the coupling constant α will be replaced by an effective, scale-dependent coupling

$$\tilde{\alpha} = \alpha[1 - (\delta - 1)\mathbb{V}], \tag{124}$$

that takes the same form as the effective running coupling computed from radiative corrections. It is remarkable that, depending on the sign of $\delta - 1$, we get either an Abelian or a non-Abelian group behavior. When $\delta < 1$, the effective charge increases with decreasing length-scale as in QED. For example, between the electron and muon scales, where only e^+e^- pairs contribute to the scale variation, the running fine structure constant reads $\alpha(\mathbb{V}) = \alpha[1 + (2\alpha/3\pi)\mathbb{V}]$, and we may make the identification:

$$\delta = 1 - \frac{2\alpha}{3\pi}. \tag{125}$$

On the contrary, when $\delta > 1$, we get an effective charge that decreases as energy increases, as in non-Abelian theories (asymptotic freedom). For example the SU(3) running coupling reads to lowest order $\alpha_3(\mathbb{V}) = \alpha_3[1 - (7\alpha_3/2\pi)\mathbb{V}]$ for 6 quarks, and we can make in this case the identification:

$$\delta = 1 + \frac{7\alpha_3}{2\pi}. \tag{126}$$

Let us now jump to special scale-relativity. The requirement of special scale covariance (leading to a reinterpretation of the Planck length-scale as a limiting, lowest scale in nature, invariant under dilations) implies introducing a generalized, scale-dependent 'fractal dimension' $D(r)$ becoming larger than 2 for scales smaller than the Compton length of the electron (equation 15). Namely, equation (120) now reads

$$\frac{< d\xi^2 >}{ds} = \frac{\hbar}{m}(ds/\lambda_e)^{(2/D(r))-1} \tag{127}$$

with $D(r) = 1 + \delta(r)$ as given by equation (15) and with $(ds/\lambda_e) \approx (r/\lambda_e)^2$. To lowest order one finds [see equation (16)]

$$\delta - 1 = \frac{1}{2}\frac{\mathbb{V}^2}{\mathbb{C}_e^2}, \tag{128}$$

with $\mathbb{V} = \ln(\lambda_e/r)$ and $\mathbb{C}_e = \ln(\lambda_e/\mathbf{\Lambda})$. Then the effective Planck 'constant' now reads to lowest order:

$$\tilde{\hbar} = \hbar[1 + \frac{1}{2}\frac{\mathbb{V}^3}{\mathbb{C}_e^2}]. \tag{129}$$

This cubic increase of the scale-relativistic correction is illustrated in Figs. 9 and 10 (where we have plotted its square root, $\approx 1 + \mathbb{V}^3/4\mathbb{C}_e^2$).

Now the variation with scale of the running charge is given by (see e.g. Ref. [1]):

$$\frac{\alpha_e}{\alpha(r)} = 1 - \frac{2\alpha_e}{3\pi}\left[\sum_{i=0}^{n} Q_i^2\mathbb{V} - \sum_{i=0}^{n}(Q_i^2\mathbb{V}_i)\right], \tag{130}$$

where α_e ($\approx 1/137.036$) is the low energy fine structure constant, n is the number of elementary pairs of fermions of dimensionless charges $Q_i = q_i/e$ and of Compton lengths $\lambda_i = \hbar/m_i c > r$, and where we have set $\mathbb{V} = \ln(\lambda_e/r)$ and $\mathbb{V}_i = \ln(\lambda_e/\lambda_i)$. This formula is written to lowest order and neglects threshold effects (see [11] for an improved treatment). Equation (130) means that the information about the masses of elementary charged fermions (contained in the Compton scales λ_i) is "coded" in the scale variation of the electric charge in terms of transitions scales where the slope abruptly changes, while the information about their charges is contained in the slopes themselves.

The remarkable result is that this scale variation of charge, up to some power $2k$, follows very closely the scale-relativistic correction (129). Namely, we find that $(\alpha(r)/\alpha_e)^k \approx [1 + (1/2)(\mathbb{V}^3/\mathbb{C}_e^2]^{1/2}$ with a remarkable precision (see Fig. 9) on the five decades containing the mass scales of elementary fermions (0.5 MeV to ≈ 100 GeV), the best fit for k covering the range 2.0 - 2.7. In Ref. [11], we attributed this cancellation to an effect of the running mass, which is related to the running charge by the relation $m/m_e = (\alpha/\alpha_e)^{9/4}$. But this relation is valid only between the electron and muon scales, so that the scale-relativistic increase in equation (120) is no longer cancelled by the variation of mass beyond the muon energy, as already recalled in Ref. [12].

Even though it is still not completely understood, it is also remarkable that this relation is precise enough for deriving from it the masses and charges of elementary fermions in terms of only the mass and the charge of the electron and the free (best-fitted) parameter k. (Recall that the mass and charge of the electron are themselves related in a more evolved version of the theory, see [13] and the following section, and that the charge can be derived from its infinite energy, bare value, see previous section). This result is illustrated in Fig. 11, in which we compare our theoretical prediction for the scale variation of the sum of the square of charges of elementary fermions to the experimentally observed variation.

Our interpretation of this result is that the electric charge is a geometric property of the electron, in the enlarged meaning of nondifferentiable, fractal geometry. The electron is itself identified with the fractal geodesics of a nondifferentiable space-time, and the virtual particle pairs that contribute to its variation are themselves manifestations of these geometric structures. In such a view, the discretization of the mass of elementary particles is a direct consequence of the quantization of charge. A more complete understanding of these relations, including the meaning of the above parameter k, can be expected from the future development of a theory of scale relativity generalized to non linear scale transformations and including motion / scale coupling (see [13] and Section 5 here for a first attempt in this direction).

Let us finally recall that this model of particle generation allowed us to make a definite prediction on the top quark mass, that was confirmed experimentally. Namely, we predict that it must be of the order and slightly higher than the W/Z mass (it has been found to have about twice their mass, 174 ± 17 GeV [40]). Indeed, the scale variation of charge beyond the W/Z energies comes no longer under QED, but under the electroweak theory. The electric charge is multiplied at high energy by a factor 8/3, and its variation becomes guided by the $U(1)_Y \times SU(2)$ group rather than by the $U(1)_{em}$ group. As a consequence, α^k and our scale-relativistic correction cross for the last time just beyond the W/Z mass: this implies a 'great desert' up to scales of the order of the GUT scale (at least with this particular mechanism).

Let us conclude this section by reminding that, in this mass generation mechanism, the electron mass and charge remain free parameters from which the other masses and charges are computed. They therefore stand as more fundamental constants, that involve a more profound level of the theory. We have seen above that

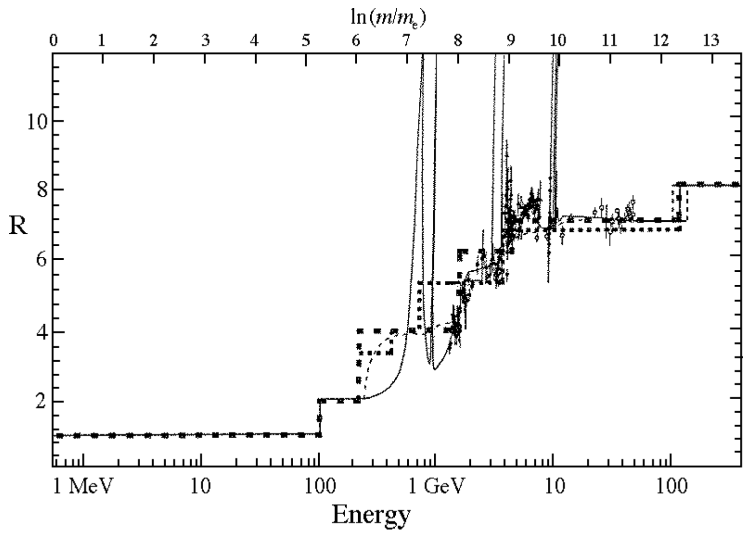

Figure 11: Comparison of our prediction for the variation with mass scale of the sum of charge squares for elementary fermions (bold broken lines) to the observed R ratio (points with error bars: experimental; thin continuous and broken line: respectively parametrization and first order QCD calculation of R, by Burkhardt et al [39]). The R ratio of the e^+e^- annihilation cross sections into hadrons and muon-antimuon pairs amounts to lowest order to the sum of the square of charges of elementary fermions. This figure (Ref. [12]) is adapted from Fig. 3 of Ref [39], by adding the leptonic e, μ, τ contributions to the hadronic ones, and by adding a top quark contribution of mass 174 ± 17 GeV [40].

the value of the low energy charge could result from a bare charge of value $1/4\pi^2$, and of its scale variation from Planck to electron scale. We shall now demonstrate that, in the framework of our new interpretation of gauge invariance and of special scale relativity, the mass of the electron is related to its charge.

6.3 Mass-charge relations

In Section 5.3, we had suggested to elucidate the nature of the electric charge as being the eigenvalue of the dilation operator corresponding to resolution transformations. We have written the wave function of a charged particle under the form:

$$\psi' = e^{i4\pi\alpha \ln(\lambda/\varepsilon)}\psi. \tag{131}$$

In the Galilean case such a relation leads to no new result, since $\ln(\lambda/\varepsilon)$ is unlimited. But if one admits that scale laws become Lorentzian below the scale λ (Section 2), then $\ln(\lambda/\varepsilon)$ becomes limited by $\mathbb{C} = \ln(\lambda/\mathbf{\Lambda})$. This implies a quantization of the charge which amounts to the relation $4\pi\alpha\mathbb{C} = 2k\pi$, i.e.:

$$\alpha\mathbb{C} = \frac{1}{2}k. \tag{132}$$

Since $\mathbb{C} = \ln(\lambda/\mathbf{\Lambda}) \approx \ln(m_P/m)$, equation (132) is nothing but a new general mass-charge relation. We have argued in Ref. [19] that the existence of such a relation could already be expected from a renormalization group approach. The existence of such mass / charge relations (that is, in an equivalent way, a Compton length scale / charge relation) is clearly apparent in Fig. 8, where we have plotted the evolution of the various inverse couplings of fundamental interactions versus scale (in the framework of the minimal, non supersymmetric standard model, made scale-relativistic): several well defined structures and symmetries relating the fundamental scales (electron, electroweak, GUT and Planck scales) to the couplings values reaveal themselves in this diagram. As we shall see, these structures are explained and accounted for by the mass / charge relation (132).

The first domain to which one can try to apply such a relation is QED. However we know from the electroweak theory that the electric charge is only a residual of a more general, high energy electroweak coupling. One can define an inverse electroweak coupling' $\bar{\alpha}_0 = \alpha_0^{-1}$ from the U(1) and SU(2) couplings (see Section 6.1):

$$\bar{\alpha}_0 = \frac{3}{8}\bar{\alpha}_2 + \frac{5}{8}\bar{\alpha}_1. \tag{133}$$

This new 'coupling' is such that $\alpha_0 = \alpha_1 = \alpha_2$ at unification scale and is related to the fine structure constant at Z scale by the relation $\alpha = 3\alpha_0/8$. It is α_0 rather than α which must be used in equation (132). Indeed, even disregarding as a first step threshold effects, we get a mass-charge relation for the electron [13]:

$$\ln \frac{m_P}{m_e} = \frac{3}{8}\alpha^{-1}. \tag{134}$$

From the known experimental values, the two members of this equation agree to 0.2 %: $\mathbb{C}_e = \ln(m_P/m_e) = 51.528(1)$ while $(3/8)\alpha^{-1} = 51.388$. The agreement is made even better if one accounts from the fact that the measured fine structure constant (at Bohr scale) differs from the limit of its asymptotic behavior. One finds that the asymptotic inverse coupling at the scale where the asymptotic mass reaches the observed mass m_e is $\alpha_0^{-1}\{r(m = m_e)\} = 51.521$, within 10^{-4} of the value of $\mathbb{C}_e$.

Now, the development of GUTs has reinforced the idea of a common origin for the various gauge interactions. Then we also expect mass-charge relations of the kind of Eq.(132) to be true for them, but at the electroweak unification scale rather than the electron one in the case of the electroweak couplings. We have suggested [13] that the following relations hold for the α_1 and α_2 couplings:

$$3\,\alpha_{1Z}\,\mathbb{C}_Z = 2; \quad 3\,\alpha_{2Z}\,\mathbb{C}_Z = 4. \tag{135}$$

From the current Z mass, $m_Z = 91.187 \pm 0.007$ GeV [41 and Refs. therein], we get $\mathbb{C}_Z = 39.7558(3)$, so that we predict $\alpha_{1Z}^{-1} = 59.6338(4)$ and $\alpha_{2Z}^{-1} = 29.8169(2)$, in good agreement with (and more precise than) the currently measured values. But, more importantly, the two relations (135) imply $\alpha_{2Z} = 2\alpha_{1Z}$, and then fix the value of the weak angle at Z scale, or, in an equivalent way, the W/Z mass ratio:

$$(\sin^2\theta)_Z = \frac{3}{13}; \quad \frac{m_W}{m_Z} = \sqrt{\frac{10}{13}} \tag{136}$$

once again in good agreement with the measured value, $(\sin^2\theta)_Z = 0.2312(4)$ [41] or 0.2306(4) for $m_t = m_H = 100$ GeV [42], to be compared with $3/13 = 0.230769$.

Note however that an uncertainty remains on the precise scale at which these relations must be written. It may be more coherent to write them at scale 123 GeV (see below), in which case scale corrections must be applied using renormalisation group equations for these various quantities. For example, $\sin^2\theta$ varies with scale beyond the Z mass as (see e.g. [1, p. 217] and references therein):

$$\sin^2\theta(m) = \sin^2\theta(m_Z) + \frac{109}{48\pi}\alpha_Z \ln\frac{m}{m_Z}. \tag{137}$$

Betwen the Z (91 GeV) and 123 GeV scales, the correction amounts to 0.0017. This yields $(\sin^2\theta)_Z = (3/13) - 0.0017 = 0.2291$, and allows us to predict a W mass value:

$$m_W = m_Z(\cos\theta)_Z = 80.06(1) GeV \tag{138}$$

that is in fair agreement (and more precise) than the current value 80.22(26) GeV [41,42]. Written at scale 174 GeV (Higgs v.e.v. and top mass), Eq. (137) would yield a correction 0.0036, $(\sin^2\theta)_Z = 0.2272$ and $m_W = 80.16(1)$ GeV.

Another possible mass-charge relation was already suggested in Refs.[1,13]. It reads $\alpha_{0\infty}\mathbb{C}_t = 1$ and, under the conjecture that the bare coupling $\alpha_{0\infty} = 1/4\pi^2$, it defines a mass scale $m_v = 123.23(1)$ GeV, that seems to be very closely related to the vacuum expectation value of the Higgs field ($174 GeV = 123\sqrt{2}$ GeV), and maybe to the top quark mass (174 ± 17 GeV).

7 SCALE RELATIVITY AND COSMOLOGY

The theory of scale relativity has not only consequence in the microphysical domain (Δx and $\Delta t \to 0$), but also in cosmology (Δx and $\Delta t \to \infty$) [1]. Some of the possible cosmological consequences are summarized in what follows, according to Refs. [16] and [17].

7.1 Large length-scales: consequences of special scale relativity

Recall that, in special scale-relativity, we first substitute to the "Galilean-like" laws of dilation $\ln\varrho" = \ln\varrho + \ln\varrho'$ the more general Lorentzian law [1,3]:

$$\ln\varrho" = \frac{\ln\varrho + \ln\varrho'}{1 + \ln\varrho \ln\varrho'/\mathbb{C}^2}. \tag{139}$$

Under this form the scale relativity symmetry remains unbroken. Such a law corresponds, at the present epoch, only to the null mass limit. It is expected to apply in a universal way during the very first instants of the universe. This law assumes that, at very high energy, no static scale and no space or time unit can be defined, so that only pure contractions and dilations have physical meaning. (One could object that the Planck scale can always be used as unit, but, as recalled hereabove, it plays a special role in scale relativity). In Eq. (139), there appears a universal purely numerical constant $\mathbb{C} = \ln\mathbb{K}$. As we shall see, the value of $\mathbb{K}$ is of the order of 5×10^{60} : its emergence yields an explanation to the Eddington-Dirac large number 'coincidences' [1,16].

Now, pure scale relativity is broken in microphysics by the mass of elementary particles, i.e., by the emergence of their de Broglie length:

$$\lambda_{\mathrm{dB}} = \hbar/mv \tag{140}$$

and in macrophysics by the emergence of static structures (galaxies, groups, cluster cores) of typical size:

$$\lambda_g \approx \frac{1}{3} Gm/ < v^2 > . \tag{141}$$

The effect of these two symmetry breakings is to separate the scale axis into three domains, a quantum (scale-dependent), a classical (scale-independent) and a cosmological (scale-dependent) domain (see Fig. 12).

The consequence is that, in the two scale-dependent domains, the static scales can be taken as reference, so that one do not deal any longer with pure dilation laws, but with a new law involving dimensioned space and time intervals:

$$\ln(\lambda"/\lambda) = \frac{\ln(\lambda'/\lambda) + \ln\varrho}{1 + \ln(\lambda'/\lambda)\ln\varrho/\ln^2(\mathbf{\Lambda}/\lambda)}. \tag{142}$$

In this new dilation law, the symmetry breaking has substituted a length-time scale that is invariant under dilations to the invariant dilation of Eq. 139. In the microphysical domain, this scale is naturally identified with the Planck scale, $\mathbf{\Lambda} = (\hbar G/c^3)^{1/2}$, that now becomes impassable and plays the physical role that was previously devoted to the zero point [1,16]. In the cosmological domain, the invariant scale is identified with the scale of the cosmological constant, $\mathbb{L} = \Lambda^{-1/2}$ [1, Chap. 7], where Λ is the cosmological constant.

We shall in this Section briefly consider the various consequences and predictions of the new theory in the cosmological domain, following closely Ref. [16] . A more detailed account can be found in Refs. [1,19].

• *Horizon / causality problem*: in the theory of scale relativity, the standard laws of dilations currently used up to now are shown to be low energy (i.e., large length-time scale) approximations of more general laws that take a Lorentzian form (see Eq. 139). As recalled hereabove, one can indeed demonstrate [1,3] that the general solution to the special relativity problem (i.e., find the laws of transformation of coordinates that are linear and satisfy the principle of relativity) is the Lorentz group. This result, which was known to apply to motion laws, applies also to scale laws (i.e., contraction and dilations of resolutions).

The horizon / causality problem is simply solved in this framework without needing an inflation phase, thanks to the new role played by the Planck lenth-time scale. It is identified with a limiting scale, invariant under dilations. This implies a causal connection of all points of the universe at the Planck epoch. The light cones flare when $t \to \mathbf{\Lambda}/c$ and finally always cross themselves (see Fig. 13 and Ref. [1]).

• *Cosmological constant and vacuum energy density:* one of the most difficult open questions in present cosmology is the problem of the vacuum energy density and its manifestation as an effective cosmological constant [43,44]. Scale relativity solves this problem and connect it to Dirac's large number hypothesis.

The first step toward our solution consists in considering the vacuum as fractal, (i.e., explicitly scale dependent). As a consequence, the Planck value of the vacuum

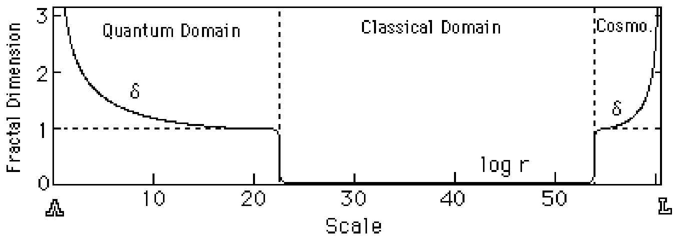

Figure 12: Variation of the fractal dimension in the three, (quantum, classical and cosmological) domains of the present era, in terms of logarithm of resolution, in the case of scale-relativistic (Lorentzian) scale laws.

energy density (that gave rise to the 10^{120} discrepancy with observational limits) is relevant only at the Planck scale, and becomes irrelevant at the cosmological scale. We expect the vacuum energy density to be solution of a scale (renormalisation group-like) differential equation [1,16,19]:

$$d\varrho/d\ln r = \Gamma(\varrho) = a + b\varrho + O(\varrho^2), \tag{143}$$

where ϱ has been normalized to its Planck value, so that it is always < 1, allowing the Taylor expansion of $\Gamma(\varrho)$. This equation is solved as:

$$\varrho = \varrho_c \left[1 + \left(\frac{r_0}{r} \right)^{-b} \right]. \tag{144}$$

We recover the well known combination of a fractal, power law behavior at small scales (here), and of scale-independence at large scale, with a fractal/non-fractal transition about some scale r_0 that comes out as an integration constant (Section 2 and Refs. [1, 11-13].

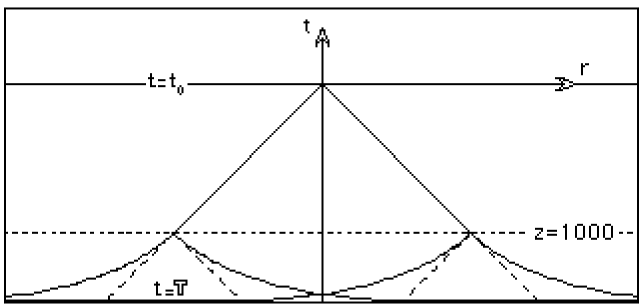

Figure 13: Illustration of the flare of light cones in scale-relativity, allowing causal connection of any couple of points in the universe (from Fig.7.1 of Ref. [1]).

The second step toward a solution is to realize that, when considering the various field contributions to the vacuum density, we may always chose $< E >= 0$ (i.e., renormalize the energy density of the vacuum). But consider now the gravitational self-energy of vacuum fluctuations [45]. It writes:

$$E_g = \frac{G}{c^4}\frac{< E^2 >}{r}. \tag{145}$$

The Heisenberg relations prevent from making $< E^2 >= 0$, so that this gravitational self-energy *cannot* vanish. With $< E^2 >^{1/2} = \hbar c/r$, we obtain the asymptotic high energy behavior:

$$\varrho_g = \varrho_P \left(\frac{\mathbf{\Lambda}}{r}\right)^6, \tag{146}$$

where ϱ_P is the Planck energy density and $\mathbf{\Lambda}$ the Planck length. From this equation we can make the identification $-b = 6$. We are now able to demonstrate one of Dirac's large number relations (see Fig. 14), and to write it in terms of invariant quantities (i.e., we do not need varying constants to implement it in this form).

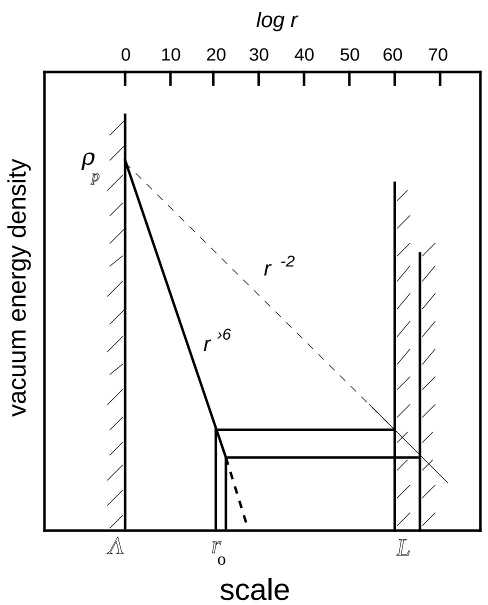

Figure 14: Variation of the gravitational self-energy density of vacuum fluctuations from the Planck length-scale to the cosmological scale $\mathbb{L} = \Lambda^{-1/2}$ (see text).

Indeed, introducing our maximal scale-relativistic length scale $\mathbb{L} = \Lambda^{-1/2}$, we get the relation:

$$\mathbb{K} = \mathbb{L}/\mathbf{\Lambda} = (r_0/\mathbf{\Lambda})^3 = (m_p/m_0)^3, \tag{147}$$

where r_0 is the Compton length of the typical particle mass m_0. Then the power 3 in Dirac's relation is understood as coming from the power 6 of the gravitational self-energy of vacuum fluctuations and of the power 2 that relies the invariant impassable scale $\mathbb{L}$ to the cosmological constant, following the relation $\Lambda = 1/\mathbb{L}^2$.

Now a complete solution to the problem would be reached only provided the transition scale r_0 be known. Consider the possible range for the cosmological constant: Ω_Λ is observationnally upper bounded by $\Omega_\Lambda < 1$. Concerning lower bounds, one may remark that any value smaller than ≈ 0.01 would be indistinguishable from zero, since it would lose any effect on the determination of the cosmological evolution. It is remarkable that the corresponding range for $\mathbb{K}$, $3 \times 10^{60} - 3 \times 10^{61}$, yields a very small range for m_0, namely 40 - 85 MeV. This short interval is known to contain several important scales of particle physics: the classical radius of the electron, that yields the e^+e^- annihilation cross section at the energy of the electron mass and corresponds to an energy 70.02 MeV; the effective mass of quarks in the lightest meson, $m_\pi/2 = 69.78$ MeV; the QCD scale for 6 quark flavours, $\Lambda_{QCD} = 66 \pm 10$ MeV; the diameter of nucleons, that corresponds to an energy 2×64 MeV. Then we can make the conjecture that the present value of the cosmological constant has been fixed at the end of the quark-hadron transition, so that the transition scale r_0 is nothing but this particular scale (Fig. 14). Along such lines, we get:

$$\mathbb{K} = (5.3 \pm 2.0) \times 10^{60}, \tag{148}$$

corresponding to $\Lambda = 1.36 \times 10^{-56}\mathrm{cm}^{-2}$ and to $\Omega_\Lambda = 0.36\, h^{-2}$ (where h is the Hubble constant in units of 100 km/s.Mpc). Such a value of Λ would solve the age problem. Indeed the age of the universe becomes larger than 13 Gyr (in agreement with globular clusters) provided $h < 0.75$ in the flat case ($\Omega_{tot} = 1$), and $h < 0.85$ if $\Omega_{tot} < 1$.

Note that all the above calculation is made in the framework of Galilean scale laws. The passage to Lorentzian laws would change only the domain of very high energies, and thus would not affect our result, since it depends essentially on what happens at scale r_0 (where the scale-relativistic corrections remain small).

•*Slope of the correlation function*: It has been observed for long that the correlation functions of various classes of extragalactic objects (from galaxies to superclusters) were characterized by a power law variation in function of scale, with an apparently universel index $\gamma = 1.8$ [46], smaller than the value $\gamma = 2$ expected from the simplest models of hierarchical formation.

The theory of scale relativity brings a simple solution to this problem. The value $\gamma = 2$ is nothing but what is expected in the framework of Galilean scale laws: it is the manifestation of a fractal dimension $\delta = 3 - \gamma = 1$. In scale relativity, one must jump to Lorentzian laws at large scales in order to ensure scale-covariance. The fractal dimension now becomes itself scale-varying and depends on the cosmological constant $\Lambda = 1/\mathbb{L}^2$ as [1]:

$$\delta = \delta(r) = \frac{1}{\sqrt{1 - \ln^2(r/\lambda_g)/\ln^2(\mathbb{L}/\lambda_g)}}, \tag{149}$$

where λ_g is the typical static radius of the objects considered (10 kpc for giant galaxies, 100 to 300 kpc for clusters...). Several consequences and new predictions arise from this formula.

First we expect $\gamma = 2$ at small scales. Several observations confirm this prediction: the flat rotation curves of galaxies imply halos in which mass varies as $M(r) = r^\delta$, with $\delta = 1$; Vader and Sandage [47] have found an autocorrelation of dwarf galaxies at small scales (10- 200 kpc) caracterized by a power $\gamma \approx 2.2$; the analysis of the CfA survey by Davis and Peebles [46] shows that, apart from fluctuations coming from deconvolution, the average γ is 2 between 10 and 300 kpc, while it reaches its value 1.8 only between 1 and 10 Mpc (see their Fig. 3).

We predict a value of 1.8 at a scale of $\approx$ 10 Mpc. Conversely, this becomes a direct measurement of the cosmological constant. We have indeed plotted in Fig. 15 the function $\gamma(r) = 3 - \delta(r)$ for various values of $\mathbb{C} = log(\mathbb{L}/\lambda_g)$, from $\mathbb{C} = 5.1$ (i.e., $\mathbb{K} = 2.3 \times 10^{60}$) to $\mathbb{C} = 6.9$ ($\mathbb{K} = 1.4 \times 10^{62}$). The best fit of the observed value of γ (1.8 at 10 Mpc) is obtained for $\mathbb{C} = 5.4 - 5.7$, i.e., $\mathbb{L} = 2.5 - 5$ Gpc, $\mathbb{K} = 4.4 - 9.3 \times 10^{60}$, $\Lambda = 1.9 - 0.18 \times 10^{-56}$ (i.e. $\Omega_\Lambda = 0.77 - 0.18$ for $H_0 = 80$ km/s.Mpc^{-1}): these values are in good agreement with our previous estimate from the vacuum energy density. This new determination is expected to be highly improved in the near future. Indeed we predict a fast variation of γ at large scales: it must fall to a value of 1.4- 1.5 at a scale of 100 Mpc for galaxies. Some recent results [48] seem to confirm such a prediction.

The transition to uniformity ($\gamma = 0$, $\delta = 3$) is reached only at very large scales ($\approx$ 1 Gpc). This seems to be confirmed by the recent suggestion that the COBE map remains characterized by a low fractal dimension $\delta = 1.43 \pm 0.07$ [49].

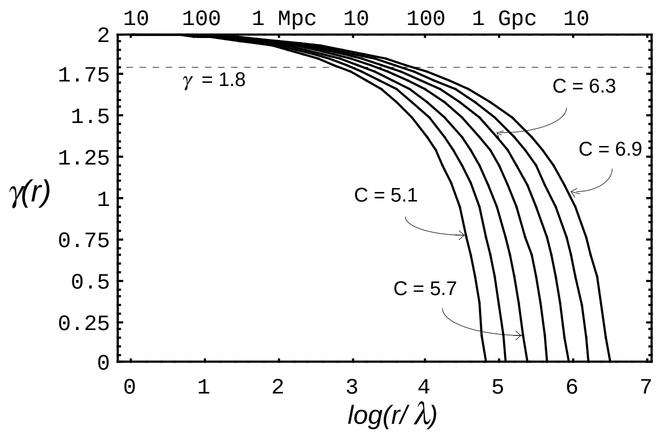

Figure 15: Variation with scale of the power of the galaxy-galaxy autocorrelation function in scale-relativity, for various values of the constant $\mathbb{C} = log(\mathbb{L}/\lambda)$, where $\mathbb{L} = \Lambda^{-1/2}$.

7.2 Large time-scales: chaotic systems

We have suggested in Ref. [1, Chap. 7.2] that the scale-relativistic methods could also be applied, as a large time-scale approximation, to chaotic systems. We shall first recall the argument, then propose a new interpretation of the physical meaning of this theory.

Consider a strongly chaotic system, i.e., the gap between any couple of trajectories diverges exponentially with time. Let us place ourselves in the reference frame of one trajectory, that we describe as uniform motion on the z axis:

$$x = 0, y = 0, z = at. \tag{150}$$

The second trajectory is then described by the equations:

$$x = \delta x_0 (1 + e^{t/\tau}),\ y = \delta y_0 (1 + e^{t/\tau}),\ z = at + \delta z_0 (1 + e^{t/\tau}), \tag{151}$$

where we have assumed a single Lyapunov exponent $1/\tau$ for simplicity of the argument. Let us eliminate the time between these equations. They become:

$$y = \frac{\delta y_0}{\delta x_0} x, \quad z = \frac{\delta z_0}{\delta x_0} x + a\tau \ln \left(\frac{x}{\delta x_0} - 1 \right). \tag{152}$$

As schematized in Fig. 16, this means that the relative motion of one trajectory with respect to another one, when looked at with a *very long time resolution* (i.e., $\Delta t >> \tau$: right diagram in Fig. 16), becomes non-differentiable at the origin, with different backward and forward slopes. Moreover, the final direction of the trajectory in space is given by the initial 'uncertainty vector' $\varepsilon^k = (\delta x_0, \delta y_0, \delta z_0)$. Then chaos achieves a kind of amplification of the initial uncertainty. But the orientation of the uncertainty vector ε being completely uncontrolable (it can take its origin at the quantum scale itself), the second trajectory can emerge with any orientation with respect to the first.

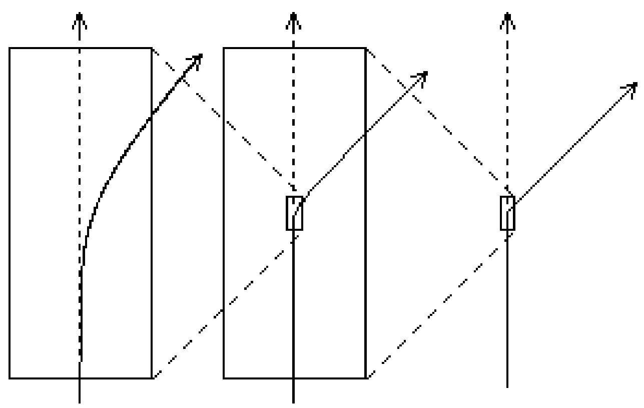

Figure 16: Schematic representation of the relative evolution in space of two initially nearby chaotic trajectories seen at three different time resolutions, τ, 10τ and 100τ (from Ref. [1]).

In the end, beyond the horizon of predictibility, the information about the trajectory at $t < 0$ has been completely lost: its description can no longer be deterministic, and we are obliged to jump to a statistical description in terms of a Markov process (independence of events). In other words, even if the basic equations remain deterministic, it is not the case of their solutions. But the behavior of the solutions is what really matters, so that one must admit that the occurrence of large time scale chaos really changes the physics.

If we now start from a continuum of different values δx_0, the breaking point in the slope occurs anywhere, and the various trajectories become describable by non-differentiable, fractal paths. In the limit, the decorelation and information loss being complete, the motion become Brownian-like, so that the fractal dimension of the trajectories becomes $D = 2$. However one must keep in mind that this is, strictly, a large time-scale approximation, since when going back to $\Delta t \approx \tau$ (left diagram in Fig. 16), differentiability is recovered.

The consequence of chaos is then twofold: the number of trajectories becomes extremely large for undistinguishable initial conditions (infinite at the limit $t/\tau \to \infty$) and the individual trajectories themselves become increasingly erratic (non-differentiable at the limit). Then the first step of our approach consists in giving up the concept of well-defined trajectory *at large time scales*, and in introducing families of virtual trajectories [1,2,14,50]. The real trajectory is one random realization among the infinite number of trajectories of the family.

The demultiplication of virtual trajectories implies to jump to a statistical description. The Brownian-like character of the motion leads one to describe the individual displacements in terms of the Markov-Wiener process $\xi(t)$ that we used in Sections 2 and 3. Moreover, the complete loss of information that happens beyond the horizon of probability ($\Delta t > 20\tau$) leads us to conjecture that time reversibility is no longer ensured for such very large time resolutions. The forward process (t increasing) and the backward process obtained by reversing the time differential ($dt \to -dt$), though remaining equivalent from the statistical point of view, are a priori characterized by different average velocities, v_+ and v_-. We are now in the same conditions as those that led us to standard quantum mechanics: doubling of velocity leading to a complex representation and fractal $D = 2$ trajectories leading to introduce new second-order terms in the differential equations.

Let us now show that a new physical meaning can be attributed to this approach. Up to now [1,14,50], we have considered the application of scale relativity to chaos as a marginal consequence of the theory, valid as a large-time scale approximation. However this point must be analysed further. We note first that, in the limit $\Delta t \to \infty$, one can no longer speak of an "approximation", since the theory becomes exact. Moreover, the transition from the classical, deterministic theory to the large time scale, nondeterministic theory is rather fast, as in the case of the classical / quantum transition. Note also that the nonfractal (i.e. scale independence) $\to$ fractal (explicit scale dependence) transition (see Fig. 12) is expected in the theory to occur toward small and large scales, for space and time resolutions. All these remarks lead us to reverse the point of view, and to consider that the universal emergence of chaos in natural systems is nothing but the manifestation of the large-scale fractal structure of space-time. The various problems implying large time resolutions become, in this view, 'cosmological' in an enlarged meaning.

7.2.1 Application to general gravitational systems

Let us briefly consider the system of equations that corresponds to a general Newtonian gravitational potential [17]. It writes:

$$\mathcal{D}^2 \Delta\psi + i\mathcal{D}\frac{\partial\psi}{\partial t} = \frac{\Phi}{2m}\psi \tag{153}$$

$$\Delta\Phi = 4\pi G \varrho_0 |\psi|^2$$

$$\mathcal{D} = \mathcal{D}(x, t, |\psi|^2)$$

This is a 'looped', highly non-linear system with feedback, in which the density of matter that intervenes in Poisson equation is precisely given by the probability density to be determined. To be fully general, we have also considered the situation where the diffusion coefficient is itself dependent on position and time, and on the local value of the probability density (see Refs. [12,14] and Section 4 here for a first treatment of the case of a variable diffusion coefficient). This is a too much complicated system to be solved in general, so that we will consider simplified situations in what follows.

However, the universal properties of gravitation allows one to reach a general statement about the behavior of this equation and its solutions. The always attractive character of the gravitational potential (except when considering a cosmological constant contribution, see below) implies that the energy of systems described by Eq. (153) will be always quantized. This equation is then expected to provide us with well defined structures in position and velocity, and then stands as a general equation for structuration of gravitational systems.

7.2.2 Quantization of the Solar System

Consider a gravitational system described by a Newtonian potential Φ such that $\Delta\Phi = 4\pi G\varrho$, and assumed to be subjected to developed chaos: assuming that one can define an average diffusion coefficient, Eq. (153) applies (as a first approximation model) to this problem. We have also seen in Section 4 that even when accounting for a slowly variable diffusion coefficient the equation still keeps the same form. Then consider a test planet (or planetesimal) orbiting in the field of the Sun, $\Phi = -GmM/r$, and describe the collective, chaotic effect of all the other planetesimals by a Brownian-like motion, as given by the double Wiener fluctuation of the above formamism. The specialization of Eq. (153) to the case of stationary motion with conservative energy $E \equiv 2i\mathcal{D}m\partial/\partial t$ yields

$$2\mathcal{D}^2\Delta\psi + [\frac{E}{m} + \frac{GM}{r}]\psi = 0. \tag{154}$$

The equivalence principle suggests that $\mathcal{D}$ is now independent of m. This equation is similar to the Schrödinger equation for the hydrogen atom, up to the substitution $\hbar/2m \to \mathcal{D}$, $e^2 \to GmM$, so that the natural unit of length (which corresponds to the Bohr radius) is:

$$a_0 = 4\mathcal{D}^2/GM. \tag{155}$$

We thus find [1,14,50] that the energies of planets scale as $E_n = GmM^2/8\mathcal{D}^2n^2$, $n = 1, 2, ...$, and that the probability densities of their distances to the Sun are confined to definite regions given by the square of the well-known radial wave functions of the hydrogen atom. We also expect angular momenta to scale as $L = 2m\mathcal{D}l$, with $l = 0, 1, ..., n-1$: this means that, unlike in quantum mechanics, E/m and L/m are 'quantized' rather than E and L. (One must be cautious that here the 'quantization' does not take as strict a meaning as in quantum mechanics: since the trajectories become classic again at small time-scales, it must be understood as indicating the occurence of preferential values, as given by the peaks of probability density and/or the average expectation values of the variables).

The average distance to the Sun and the eccentricity e are given, in terms of the two quantum numbers n and l, by the following relations:

$$a_{nl} = [\frac{3}{2}n^2 - \frac{1}{2}l(l+1)]a_0 \tag{156}$$

$$e^2 = 1 - \frac{l(l+1)}{n(n-1)}.$$

Let us now briefly compare these predictions to the observed structures in the Solar System. Note that the difference of physical and chemical composition of the inner and outer solar systems suggests to us that they can be treated as two different systems, i.e., that we expect two different diffusion coefficients for them. [See Sec. (iii) below for a possible justification of this point]. The main results are summarized hereafter (see Fig. 17).

(i) *Distribution of eccentricities of planets*: the observed orbits of the planets in the solar system are quasi-circular. Even the largest eccentricities (Pluto, $e^2 = 0.065$; Mercury, $e^2 = 0.042$) actually correspond to small values of e^2. Such a result is clearly a prediction of our theory: Indeed, equation (156b) implies that, after the purely circular state $l = n-1$, the first non circular state, $l = n-2$, yields eccentricities larger that 0.58 for $n \leq 6$ (which is the range observed for n in the solar system, see below). Such a large value would imply orbit crossing between planets and strong chaos and cannot correspond to a stable configuration on large time scales. Then only the quasi-circular orbits remain admissible solutions. Such a conclusion is relaxed in the case of comets and asteroids, and it could be interesting to compare our prediction to their distribution. This will be done in a forthcoming work.

(ii) *Distribution of planet distances.* we may now compare the observed values of semi-major axes of the planets to our prediction (156a) with $l = n-1$: $\sqrt{a} = n(1+1/2n)^{1/2}\sqrt{a_0}$, for the inner and outer systems respectively. Note that the ordinate at origin is predicted to be zero. This prediction is very well verified for the two systems: we find $a_{\rm int}(0) = 2 \times 10^{-4}$ A.U. and $a_{\rm ext}(0) = 4 \times 10^3$ A.U..

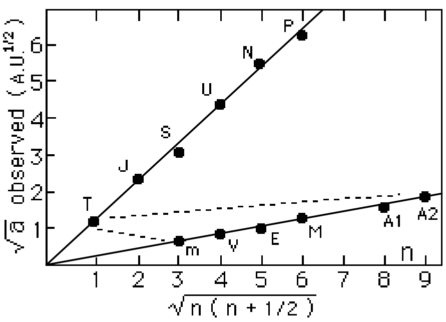

Figure 17: Comparison of the observed average distances of planets to the Sun with our prediction (see text). The abscissa is labelled by the value of n, but is given by $\sqrt{n(n+1/2)} \approx n + 1/4$. A1 and A2 are for the two main peaks in the distribution of asteroids in the asteroid belt. 'T' stands for the whole inner solar system (telluric planets), which corresponds to 'orbital' $n = 1$ of the outer system. A new possible small planet at $n = 2$ of the inner system is predicted (distance to the Sun : 0.185 A.U.).

Mercury, Venus, Earth and Mars take respectively ranks $n = 3$, 4, 5 and 6 in the inner system. The average slope is $(\sqrt{a_0})_{\rm int} = 0.195 \pm 0.0022$. One of the most fascinating result of this theory is the fact that there are two more orbitals than actually observed. Can intramercurial planets survive, and if so, how could they have escaped observation ?

The orbits $n = 1$ corresponds to a distance so close to the Sun (0.05 A.U.), that its emptiness may be easily understood: the temperature would reach 2000 K and it seems difficult for a telluric body to survive in these conditions. On the contrary, the possibility that the orbital $n = 2$ hosts a still undiscovered small planet is not excluded. Indeed the temperature would be about 900 K, which would not prevent silicates to survive, provided the planet is massive enough. The general relativistic constraint on the advance of perihelon of Mercury still allows the existence of a small planet ≈ 1500 times less massive than the Earth without destroying the agreement of theory with observations. Such an object would be both as massive as the largest asteroids in the asteroid belt and small enough to have escaped any discovery by visual detectors. A project of detection in the infra-red is now scheduled [51].

The central peak of the asteroid belt (2.7 A.U.) agrees remarkably well with $n = 8$ of the inner system, and the main peak (3.15 A.U.) with $n = 9$. Including them yields $\sqrt{a_0}_{\rm int} = 0.195 \pm 0.0017$. This result may help understanding the fact that there is no large planet there: the zone where the belt lies, even though it corresponds to maxima of probability density for the inner system, also corresponds to a minimum in the outer system. The region between Mars and Jupiter is where the two systems overlap. The emptiness of the orbits $n = 7$ and $n = 10$ is easily understandable, since they coincide with the resonances 1:4 and 2:3 with Jupiter, where small timescale dynamical chaos is expected to occur [52].

Jupiter, Saturn, Uranus, Neptune and Pluto rank $n = 2, 3, 4, 5, 6$ in the outer system (see Fig. 17). The average slope is $(\sqrt{a_0})_{\rm ext} = 1.014 \pm 0.016$. The average distance of the inner solar system in very good agreement with $n = 1$ of the outer system (see below our suggestion that it corresponds to a secondary process of fragmentation): including it yields an improved slope $(\sqrt{a_0})_{\rm ext} = 1.014 \pm 0.012$. Note also the agreement of Neptune and especially Pluto with the outer relation (recall that they did not fit the original Titius-Bode law).

(iii) *Distribution of mass in the solar system.* : not only the distribution of planet positions, but the distribution of mass itself is not at random in the solar system. Consider first the outer system, in which the average inner system is counted as one. We see the mass increase, reach a peak with Jupiter, then decrease up to Pluton (this decrease may continue with the possible ultra-plutonian small bodies). Now consider the inner solar system: the mass distribution follows the same shape, with an increase for Mercury to Earth, then a decrease up to the asteroids.

Such a mass distribution is in agreement, at least in its great lines, with the laws of probability density derived from equation (154), which writes for the various values of n (circular orbits, $l = n - 1$, and $\int P(r)dr = 1$):

$$P(r) \propto \frac{1}{2n!}(\frac{2}{na})^{2n+1} r^{2n} e^{-2r/na}. \tag{157}$$

This suggests to us a possible mechanism for the mass distribution in the solar system.

The first step would be a distribution of planetesimals according to the fundamental orbital ($n_0 = 1$), which is in qualitative agreement with the global mass distribution of the present planets. Then a first process of fragmentation would occur, once again according to equation (153), in which the potentiel would be given by the Poisson equation for the density in this orbital. The peak of probability density will give rise to the formation of the most massive planet in the system, i.e. Jupiter, which fixes the unit in equation (154) and for all other length scales. The remaining planetesimals would then make the other planets of the outer system (see Fig. 18), with distances increasing in terms of a new index n_1. However, although far from the Sun the planetesimals accrete in only one planet in each orbital, tidal effects imply for the fundamental one ($n_1 = 1$) a new fragmentation process in terms of a third 'quantum number', n_2. This 'orbital' is then identified with the whole inner solar system. The advantage of such a process is that it relates the scales of the inner and outer systems and then reduces the number of free parameters to only one. Indeed the ratio of distances between the peak of orbital $n_1 = 2$ (Jupiter) and the peak of orbital $n_1 = 1$ (which is identified with the planet of largest mass in the inner solar system, i.e. the Earth) is expected to be $a_J/a_E \approx 5$, in good agreement with the observed value 5.2.

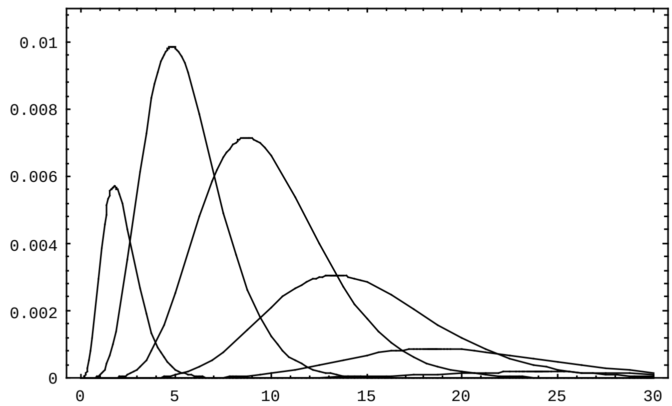

Figure 18: Possible mechanism of fragmentation for the mass distribution in the solar system (see text).

(iv) *Distribution of angular momentum.* our finding that L/m is quantized rather than L allows us to suggest a solution to the problem of the distribution of angular momentum among planets (Jupiter 60 %, Saturn 25 %). Indeed, since the quantum number n remains small (≤ 6), the distribution of angular momentum is expected to mainly mirror that of mass: then most angular momentum must be carried on by the largest planets, as observed.

7.2.3 Quantization of galaxy pairs

The application of our scale-relativistic method to the problem of galaxy pairs is quite similar to the solar system case [17]. We start from the remark that even an 'isolated' pair in galaxy catalogues is never truly isolated. We shall then describe the effect of the uncontrolable interactions of the environment in terms of the hereabove complex Wiener process.

In classical as well as quantum mechanics, the problem of the relative motion of two bodies can be reduced in the reference system of the center of inertia to that of one body of mass:

$$m = \frac{m_1 m_2}{m_1 + m_2}. \tag{158}$$

In the particular case of gravitation, the potential is $\phi = -Gm_1 m_2/r$, so that the equation of motion is that of a test particle around a body of mass $M = m_1 + m_2$. The same is true here, so that equation (154) still applies to this case.

We find that the pair energy is quantized as:

$$E_n = -\frac{1}{2} m \left(\frac{GM}{2\mathcal{D}n}\right)^2 , \tag{159}$$

and that the relative velocity in binary galaxies must take only preferential values given by:

$$v_n = \frac{GM}{2\mathcal{D}n} = \frac{v_1}{n}. \tag{160}$$

Such a theoretical result seems to provide an explanation for Tifft's effect of redshift quantization in binary galaxies. Indeed it has been claimed by Tifft [53] that the velocity differences in isolated galaxy pairs was not distributed at random, but showed preferential values near 72 km/s, 36 km/s and 24 km/s, i.e. $72/n$ km/s, with n = 1, 2, 3,... . This result, in particular the 72 km/s periodicity (see Fig. 19), was confirmed by several authors (see, e.g., [54,55]), and can actually be seen in practically any sample of galaxy pairs with high quality velocities.

However a global quantization with nearly the same velocity differences, 72 km/s [56,57] and 36 km/s [58,59]), has also been found in samples of nearby galaxies, even when pairs are excluded. We shall now see how this effect can also be understood in our framework, but in terms of a cosmological effect, then recall briefly how the 'global' quantization and the pair quantization must be related.

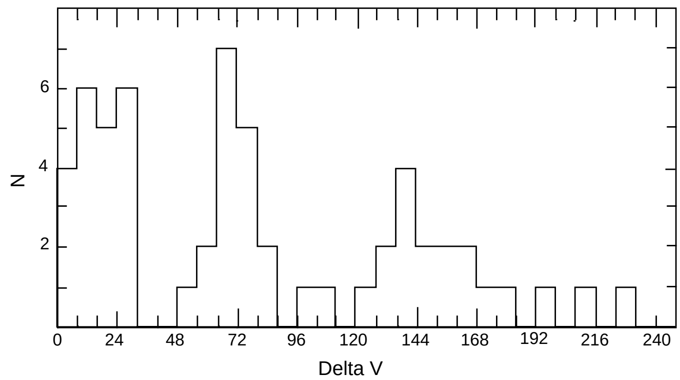

Figure 19: Velocity difference between members of galaxy pairs (from Cocke, Ref. [55]). Note the peaks at 72 km/s, 144 km/s and possibly 24 km/s.

7.2.4 Quantization in cosmological background

Start with the Robertson-Walker form of the metric of Friedman-Lematre models:

$$ds^2 = c^2dt^2 - R^2(t)\left\{\frac{dX^2}{1-kX^2} + X^2(\sin^2\theta d\varphi^2 + d\theta^2)\right\}, \tag{161}$$

where $R(t)$ is solution of the standard equations of cosmology, that can be reduced to the Einstein equation,

$$\dot{R}^2 + k = \frac{8\pi}{3}G\varrho R^2, \tag{162}$$

the energy-conservation equation,

$$\frac{d}{dR}(\varrho R^3) + 3pR^2 = 0, \tag{163}$$

and the thermodynamical equation of state that writes for a dust model (that we shall only consider here):

$$p = 0. \tag{164}$$

The metric can also be written under the form:

$$ds^2 = c^2dt^2 - R^2(t)\{d\chi^2 + S^2(\chi)(\sin^2\theta\, d\varphi^2 + d\theta^2)\}, \tag{165}$$

where $S(\chi) = [\sin\chi, \chi, \sinh\chi]$ for $k = [1, 0, -1]$ respectively. We shall call the inverse function $\mathrm{ArcS}(\chi) = [\mathrm{Arcsin}\chi, \chi, \mathrm{Argsinh}\chi]$.

Consider now the motion of a body, and let us define a distance as:

$$r = R(t)\chi. \tag{166}$$

For a comoving body, $\chi =$ constant, so that the 'velocity' corresponding to such a distance is:

$$v = \dot{r} = \dot{R}\chi = \frac{\dot{R}}{R}(R(t)\chi) = Hr. \tag{167}$$

In terms of this distance, the Hubble law keeps its simplest linear form (but with $H = H(t)$). This distance is nothing but the well-known proper distance, whose expression in terms of redshift writes:

$$r = \frac{R_0}{1+z}\,\mathrm{ArcS}\left(\frac{D_L}{R_0}\right) \tag{168}$$

where D_L is the luminosity-distance:

$$D_L = \frac{c}{H_0 q_0^2}\,\{q_0 z + (q_0 - 1)[-1 + (1 + 2q_0 z)^{1/2}\}. \tag{169}$$

The fundamental equation of dynamics writes in terms of the proper distance:

$$\frac{d^2}{dt^2}r = -\nabla\left(\frac{2\pi}{3}G\varrho r^2\right). \tag{170}$$

The application of our method to cosmology is now very simple, since the potential in equation (170) is that of the three-dimensional harmonic oscillator when $\varrho = cst$, which is one of the best studied potential in quantum mechanics. Strictly, ϱ is time-dependent, but we can neglect this time-dependence as a first approximation. Note also that, since the problem that we treat here is simply that of a uniform density, our results will apply to more general situations different from the cosmological one, in particular to the problem of emergence of structures inside bodies (for example, inside galaxies), in the case where a uniform density can be considered as a good approximation of the actual density.

We thus assume that a particle of the cosmological fluid is also subjected to a continuous, uncontrolable action of its environment, that we describe by the twin-Wiener process with diffusion coefficient $\mathcal{D}$ introduced hereabove. This amounts to apply our quantization method to equation (170). The equation for the probability amplitude of a "test particle" becomes:

$$\mathcal{D}^2\Delta\psi + i\mathcal{D}\frac{\partial}{\partial t}\psi - \frac{\pi}{3}G\varrho r^2\psi = 0. \tag{171}$$

Let us look for stationary solutions of the equation, which can now be written as

$$\mathcal{D}^2\Delta\psi + \left[\frac{E}{2m} - \frac{\pi}{3}G\varrho r^2\right]\psi = 0. \tag{172}$$

Up to the substitution

$$\hbar \to 2m\mathcal{D}, \tag{173}$$

this is the Schrödinger equation for a 3-dimensional isotropic harmonic oscillator with frequency

$$\omega = \left(\frac{4\pi G\varrho}{3}\right)^{1/2}. \tag{174}$$

The solution is well-known and can be found in any textbook on quantum mechanics. The energy is quantized as:

$$E = 4m\mathcal{D}\left(\frac{\pi G\varrho}{3}\right)^{1/2}\left(n + \frac{3}{2}\right), \tag{175}$$

where $n = n_1 + n_2 + n_3$, the n_i corresponding to three linear harmonic oscillators for the three coordinates. The probability density of the stationary states can be written as:

$$|\psi_{n_1n_2n_3}|^2 \propto e^{-(r/a)^2}[\mathcal{H}_{n_1}(x/a)\mathcal{H}_{n_2}(y/a)\mathcal{H}_{n_3}(z/a)]^2 \tag{176}$$

where the $\mathcal{H}_n'$ s are the Hermite polynomials, and where a is a characteristic length scale given by:

$$a = \sqrt{2\mathcal{D}/\omega} = (\mathcal{D})^{1/2}(\pi G\varrho/3)^{-1/4}. \tag{177}$$

Recall that the first Hermite polynomials are:

$$\mathcal{H}_0 = 1;\ \ \mathcal{H}_1 = 2x;\ \ \mathcal{H}_2 = 4x^2 - 2;\ \ \mathcal{H}_3 = 8x^3 - 12x; ... \tag{178}$$

We then predict that, when the average density is ϱ, matter will have a tendency to form structures according to the various modes of the quantized 3-D harmonic oscillator as given in equation (176). The zero mode is a Gaussian of dispersion $\sigma_0 = a/\sqrt{2}$. The mode $n = 1$ is a binary structure whose peaks are situated at

$x_{\text{peak}} = \pm a$. The mode $n = 2$ has three peaks at 0, $\pm\sqrt{5/2}\,a \approx \pm 1.58\,a$. For $n = 3$, one finds $x_{\text{peak}} = \pm 0.602\,a$ and $\pm 2.034a$. More generally, the position of the most extreme peak can be approximated by the formula

$$x_{max} = (n^2 + 3n)^{1/2}\,\frac{a}{2} \approx (n + \frac{3}{2})\frac{a}{2}. \tag{179}$$

If one now considers the momentum representation rather than the position one, one predicts a distribution of velocities that is given by exactly the same functions, but with a replaced by the characteristic velocity:

$$v_0 = \sqrt{2\mathcal{D}\omega} = 2\mathcal{D}^{1/2}(\pi G\varrho/3)^{1/4}. \tag{180}$$

From the hereabove study, we note that the difference between the extreme velocity peaks is of the order of $\approx 2v_0$, $\approx 3v_0$, and $\approx 4v_0$ for the modes $n = 1,\ 2,\ 3$ respectively.

The main conclusion of this Section is that we predict that the various cosmological constituents of the universe will be situated at preferential relative positions and move with preferential relative velocities, as described by the various structures implied by the quantization of the harmonic oscillator. In other words, we expect the Universe to be locally structured, in position and velocity, according to the SU(3) group, which is the dynamical symmetry group of the isotropic three-dimensional harmonic oscillator.

Much work is needed to compare the available data with such a prediction. Let us only remark here that the linear-like quantization of the harmonic oscillator case yields a remarkable explanation to the 'global' quantization in units of 36 km/s found by Tifft, Guthrie and Napier and others [56-59]. Moreover, one can demonstrate, by treating the case of a potential which is Keplerian at small scale and harmonic at large scale [17], that the 'Kepler' and 'harmonic' quantizations must be related: this explains the observed relation between the two quantizations (72 and 36 km/s).

7.2.5 Dissipative systems: first hints

One can generalize the Euler-Lagrange equations to dissipative systems thanks to the introduction of a dissipation function $\mathcal{F}$ (see e.g. [60, p.107]):

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\mathcal{V}_i} = \frac{\partial\mathcal{L}}{\partial x_i} - \frac{\partial\mathcal{F}}{\partial\mathcal{V}_i}, \tag{181}$$

where $\mathcal{F}$ is linked to the energy dissipation by the equation $\mathcal{F} = -d\mathcal{E}/2dt$. This becomes in the Newtonian case:

$$m\frac{d}{dt}\mathcal{V}_i = -\nabla_i\Phi - \frac{\partial\mathcal{F}}{\partial\mathcal{V}_i} = -\nabla_i\Phi - \Sigma_j k_{ij}\mathcal{V}_j. \tag{182}$$

Consider only the simplified isotropic case:

$$f = kv, \tag{183}$$

and its complex generalization:

$$\mathcal{F} = k\mathcal{V}. \tag{184}$$

We obtain a new generalized equation:

$$\mathcal{D}^2\Delta\psi + i\mathcal{D}\frac{\partial\psi}{\partial t} - \frac{\Phi}{2m} + i\frac{k}{m}\psi\ln\psi = 0 \tag{185}$$

which is still Schrödinger-like, since it corresponds to a perturbed Hamiltonian: $H = H_0 + V$, with the operator V such that $V\psi = -i(k/m)\psi\ln\psi$. The standard methods of perturbation theory in quantum mechanics can then be used to look for the solutions to this equation. This will be presented in a forthcoming work.

8 DISCUSSION

Before concluding this review paper, we want to discuss two points particularly linked to the theme of the present volume, namely the diffusion approach to quantum mechanics and the interpretation of Young hole experiments. We shall in particular consider the case of a recent version of this experiment that includes "quantum erasing" of the which-way information. Then we shall briefly discuss the relations of our scale-relativistic approach to other possibly equivalent or complementary ones.

8.1 Diffusion interpretation of the theory

Even though our theory is not a diffusion theory in its essence, its interpretation in terms of diffusion plays a central role in its understanding, and also for its future generalizations. But it must be clear that, even if one of the effects of the fractal and nondifferentiable structure of space-time is to "diffuse" the various geodesics, the properties of this diffusion process are not that of standard diffusion: for example, one finds on typical solutions that it may even sometimes "focalize" the virtual geodesics instead of dispersing them [19]. We shall consider in what follows only the simplest case (Galilean scale relativity, D = 2) that leads to standard quantum mechanics.

To recover a diffusion interpretation, the first step, as recalled in Section 4.1, consists in taking the imaginary part of our generalized Schrödinger equation (52). This yields the equation of continuity (54):

$$\partial\varrho/\partial t + \mathrm{div}(\varrho V) = 0, \tag{186}$$

Taking the imaginary part of equation (46), we obtain for the imaginary part of the complex velocity, ($\mathcal{V} = V - iU$,one of Nelson's equations [26, 27]:

$$U = \mathcal{D}\Delta\ln\varrho, \tag{187}$$

where $\mathcal{D} = \lambda/2$ is interpreted in this case as a diffusion coefficient. This implies $\mathrm{div}\{\varrho[U\mathcal{D}\Delta\ln\varrho]\} = 0$. This equation, combined with the continuity equation, becomes a complex Fokker-Planck equation:

$$\partial\varrho/\partial t + \mathrm{div}(\varrho\mathcal{V}) = -i\mathcal{D}\Delta\varrho. \tag{188}$$

Now reintroducing the forward and backward mean velocities, we can demonstrate the backward and forward Fokker-Planck equations which are at the basis of Nelson's stochastic quantum mechanics:

$$\partial\varrho/\partial t + \mathrm{div}(\varrho v_+) = \mathcal{D}\Delta\varrho, \tag{189}$$

$$\partial\varrho/\partial t + \mathrm{div}(\varrho v_-) = -\mathcal{D}\Delta\varrho.$$

Adding to these equations the identification of the basic fractal fluctuation (Eq. 30), $< d\xi^2 >= 2\mathcal{D}dt$, with the expression for a Markov-Wiener process, we have now completed the elements needed to obtain a diffusion, Brownian motion-like interpretation of our equations. Recall however that this is not standard diffusion, since, rather that combining a forward Kolmogorov equation (i.e., Fokker-Planck), and a backward Kolmogorov equation, equations (189) are two Fokker-Planck equations (the standard forward and a backward obtained by inversing the sign of time).

We stress once again the fact that diffusion here is only an interpretation. Our theory is not statistical in its essence, contrarily to quantum mechanics or to diffusion approaches. In scale relativity, the fractal space-time can be completely 'determined', while the undeterminism of trajectories is not set as a founding stone of the theory, but as a consequence of the nondifferentiability of space-time. In our theory, 'God does not play dice', in the sense that the fundamental laws of nature are not probabilistic, but instead probabilities come as consequences of definite laws, as was demanded by Einstein in his requirement for realism.

8.2 Quantum eraser and which-way information in the geodesics interpretation

We have already demonstrated in detail in previous publications that the Young double slit experiment, even in its more sophisticated versions imagined by Feynman, can be fully and simply understood in terms of our fractal space-time geodesics interpretation (see Refs. [61], [2] Sec. 6.4, [1] Chap. 5.5 and Fig. C8). Scully, Englert and Walther (SEW) [62] have recently proposed a new version of this experiment in which (i) one can know by which slit the 'particle' passed without disturbing its wave function in any manner, and (ii) the 'which-way information' could be erased *after* the arrival of particles on the screen, thus allowing to retrieve an interference pattern that is hidden in an apparently non-fringy pattern. We shall discuss here again the geometrical geodesics interpretation of quantum mechanics, and show how it yields a natural explanation of this new thought experiment and draws attention to El Naschie's fractal analogy given in [73] which led to the same conclusion as [62].

In the SEW experiment, atoms are used instead of photons or electrons. They are first excited by a laser beam. This excitation changes their internal level of energy, but not their kinetic energy or momentum. Then they are desexcited in a cavity by emitting a single photon, and they recover their initial state before passing through one of the two slits. One can keep the photon in the cavity and eventually change the cavity geometry after arrival of the atom on the screen in such a way that one can decide to detect the photon, then keeping the which-way information (this leading to no interference pattern), or not (this allowing to recover an interference pattern by correlation of photons and atoms). The interest of such an experiment is to demonstrate in a definitive way that its interpretation cannot be done in terms of Heisenberg inequalities and measurement theory (i.e., the interference pattern would disappear because of the uncontrolable perturbation due to the measurement device). Scully at al. then suggest to interpret it in terms of Bohr's principle of complementarity.

It is clear that our interpretation of quantum mechanics in terms of geodesics of a fractal space-time allows to explain the SEW experiment in a very simple, nearly obvious way. But our explanation refers neither to Bohr's complementarity nor to Heisenberg's uncertainty relations. It is actually very close to Feynman's probability amplitude viewpoint, since it corresponds to a geometric achievement of it (so that our predictions will be the same as that of quantum mechanics). In our view, the complex probability amplitude describes in a global way the collective properties of the beam of potential geodesics that connect the source to the screen.

As was understood by Feynman, most of the 'mystery' of quantum mechanics is contained in the existence of the probability amplitude and in its nature of complex number, i.e., of being defined by two quantities (say, a module and a phase) rather than one classically. We recall once again that such an information doubling is predicted in our theory and leads to the construction of a complex wave function (see Section 3). Indeed, giving up the hypothesis of nondifferentiability of space-time implies a local breaking of time reflection invariance, then to replace the classical velocity v by two mean velocities $[v_+, v_-\ ]$, and more generally the classical time derivative by *two* backward and forward mean derivatives. Complex numbers achieve the simplest representation of such a doubling of information [19]. The wave function is nothing but another expression for the action, that has itself become complex.We have seen that our geodesics equation $d^2x/dt^2 = 0$ can be integrated in terms of our complex action $\$ = -i\hbar\nabla \ln \psi$ to yield:

$$(\frac{1}{2}\lambda^2\Delta + i\lambda\frac{\partial}{\partial t} - \frac{\Phi}{m})\psi = 0, \tag{190}$$

which is the Schrödinger equation when $\lambda = \hbar/m$. It is clear from this equation that, if ψ_1 and ψ_2 are solutions, $\psi = \psi_1 + \psi_2$ is also a solution. Now, we have constructed Eq.(190) from the very beginning as an equation acting on average properties of an infinite ensemble of geodesical curves and we have finally found that it is linear in terms of ψ. Therefore the superposition principle follows, and also follows the rule that, if an event can occur in two alternative ways, the probability amplitude is the sum of the probability amplitudes for each way considered separately.

Then any experiment in which the 'which-way information' can be known amounts to *sort out* the geodesics, i.e., it implies to make probabilistic predictions using only the beam of geodesics that pass through one of the slit and is so described by either ψ_1 or ψ_2. If it cannot be known, each individual particle will follow at random one of the geodesics among the family of geodesics that pass through the two slits, and that is described by $\psi_1 + \psi_2$, (see Fig. C8 of Ref. [1]). But it must be clear that each fractal geodesic passes through one slit or the other, never through both: this can be checked by Feynman's path integral formulation, that shows that such trajectories, being too far from the classical trajectory, would be destroyed by destructive interferences. It must also be clear that our interpretation of such a non-relativistic experiment does not need introducing segments of trajectories that run backward in time. In our approach, such segments must be introduced only at scales smaller than the Compton length of the particle (below which time itself becomes fractal), since they manifest themselves in terms of virtual particles and radiative corrections. For atoms, the Compton length and its associated Einstein-deBroglie time $\tau = \hbar/mc^2$ are extremely small with respect to the typical scales of the experiment. Our interpretation differs from El Naschie's [63] whose fractal DNA-like informational field is, of course, non-local due to its dust-like transfinite Cantorian set nature. Finally, that the 'which-way information' be known or erased *after* the arrival of the atoms on the screen does not change anything to the problem.

We think that the apparent problem comes precisely from the belief in complementarity (the particle is either a 'corpuscle', or a 'wave', but never both simultaneously), and in the claim that the one-slit case shows a 'particle' behavior and the two-slit case (interference pattern) a 'wave' behavior.

In our opinion, such a belief is in contradiction with the formalism of quantum mechanics itself (and then automatically with our scale relativity approach, since we fully recover this formalism). Indeed, this formalism (and its success in making definite predictions) tells us that the information is doubled with respect to classical physics, since correct predictions can be made only assuming that a *complex* probability amplitude $\Psi = P\, e^{i\theta}$ is carried on from the source to the screen (this coming in scale relativity from a breaking of time reflexion invariance). This means that the 'wave' nature of the particle is present in both cases, one-slit and two-slit. That we never directly observe the phase θ is only an *experimental* limitation: even in the two-slit case, we do not see the phase! Indeed, if $\Psi_1 = \sqrt{P}\, e^{i\theta_1}$ and $\Psi_2 = \sqrt{P}\, e^{i\theta_2}$, in each one-slit experiment, we get in the two-slit case:

$$\Psi = \sqrt{2P}[1 + \cos(\theta_2 - \theta_1)]e^{i\theta}. \tag{191}$$

The phase information *of the one-slit case* is sent into the probability *module* of the two-slit case, while the phase θ of the two-slit case is still unobserved (it could be observed by performing self-interferences on the resulting wave $\Psi = \Psi_1 + \Psi_2$). So the interference pattern of the two-slit experiment finally shows and then demonstrates the wave nature (i.e., the existence of a phase) of the particle in the one-slit case.

The particular case of the SEW experiment can easily be recovered and understood also from the same formula. Indeed, the probability density in (Eq. 191) can be written as $4P\cos^2(\Delta\theta/2)$. Under a simple phase change it becomes $4P\sin^2(\Delta\theta/2)$. It is then quite possible to have an apparently nonfringy pattern that is nothing but the sum of two interference patterns, $4P[\cos^2(\Delta\theta/2) + \sin^2(\Delta\theta/2)] = 4P$, as in the 'quantum eraser' situation.

More profoundly, we think that most of the discussion about Young double-slit-like experiments is a mere consequence of attempts to still understand the quantum behavior in terms of classical concepts, and in the end to reduce it to such classical concepts, while it may actually be irreducible to them. As well 'particle' as 'waves' are classical concepts that have been built from the observation of macroscopic objects. The viewpoint that is developed in the scale-relativity approach is that microscopic objects are neither 'particle' nor 'waves', but that their behavior is that of the (fractal) geodesics of a nondifferentiable space-time. The nondifferentiable behavior is irreducible to differentiable processes, and is thus fundamentally non-classical. A similar entity is considered by El Naschie to be his Cantorions [8,63,73].

There is no 'particle', since this concept would mean that some massive point owning internal properties such as spin and charge follows one of the (undeterministic) virtual geodesics, or, in the Copenhagen interpretation, is subjected to the probability amplitude. But we have shown [1,2,13] that the mass, the spin and the electric charge can be described as geometric properties of the fractal trajectories, (and we make the conjecture, as a working hypothesis for the future developement of the theory) that the same will be true of the other internal quantum numbers.

The ‘particle’ behavior is, in this framework, a manifestation of the particular role played by extremal curves of topological dimension 1 (i.e., geodesics) in a space-time theory, and by extension by localized beams of such geodesics.

There is no ‘wave’, since this classical concept actually relies on the properties of an underlying medium. No such medium is needed, since the ‘wave’ properties (i.e., the existence of a phase) can be recovered from beams of trajectories and /or from stochastic or diffusion process, as has been well understood by Nelson [26,27], Boyarski and Gora [64] and El Naschie’s Cantorian quantum sets and the four points chaos game [73,74].

In summary, our point of view is that quantum objects are neither ‘waves’ nor ‘particles’ , but are instead, always and simultaneously, characterized by the module and the phase of their probability amplitude, while our experiments, being incomplete, put into evidence only the module. There is no ‘complementarity’ here, since the phase is never directly seen, but only indirectly through its possible appearance in the module (probability) term, and since the full, doubled information (probability and phase) is always present, even if we observe only half of it. There is therefore no mystery when one can jump instantaneously from observing the ‘wave’ behavior to observing the ‘particle’ behavior without physically disturbing the system, but only by changing the observing way. Both properties were present before the observation, even if only one of them was seen.

8.3 Other approaches

Let us conclude this section by briefly quoting (in a non exhaustive way) other approaches to the quantum and chaos problems, that are possibly equivalent or complementary to ours. This field is indeed now experiencing a fast expansion, since it becomes now clearer and clearer that fractals are not only useful in building models for natural phenomena, but stand out as a new and powerful tool for the construction of fundamental theories. Such ideas have been developed in particular by Ord (fractal space-time model and generalization of Feynman’s chessboard model [7,37,65]); Rössler (chaos explanation of quantum behavior [66]); Prigogine and Petrosky (complex, quantum-like equations describing classical chaos [67]); El Naschie (Cantorian DNA-like space-time and diffusion equations and Banach-Tarski theorem [8,63,73-75]); Kröger (fractal paths in solid state physics, nuclear matter and Quantum Field Theories [68,69]); Boyarski and Gora (model of structured space-time implying interferences of particles [64]); Le Méhauté (new electromagnetic properties in fractal media [24,70]); Dubrulle et Graner (generalized scale-invariant approach to turbulence [71]), and Castro (strings in the framework of the special theory of scale-relativity [72]).

9 SUMMARY AND CONCLUSION

As a conclusion of this review, let us achieve one of the aims of the present contribution, that is to give a summary of the various results and theoretical predictions of the new theory. Since the consequences of scale relativity cover a wide range of physical domains, these results and predictions were up to now dispersed in different papers written for different communities. This review paper is a good occasion to collect them (in a not fully exhaustive way, since some recently obtained results are still in preparation [19]), and thus to provide the reader with a wider view of the abilities of the theory.

Let us first remark that the various results of a theory may be classified according to different 'levels' :

(i) There are 'conceptual' results, namely contributions of a theory in understanding previously misunderstood general facts or in solving general problems (for example, in our case, understanding of the origin of the complex nature of the wave function; reconciling quantum physics with the relativistic approach).

(ii) There are numerical, quantified results, i.e., theoretical predictions of already measured quantities that had still no theoretical explanation (for example, prediction of the GUT and electroweak scales in particles physics, prediction of the value of the power of the galaxy-galaxy autocorrelation function in cosmology).

(iii) There are finally pure theoretical predictions, either of new still unobserved phenomena, or of the still unknown value of measurable quantities. These 'blind' predictions play a special role in testing a theory, since they are the key to its falsifiability (for example, our prediction of new planets in the solar system, or of the value of the cosmological constant).

Note that some results may fall in two or three of these items, since a numerical theoretical prediction may agree with some already measured experimental result, but remain more precise. The blind prediction is only about the additional unknown figures in this case (example: our prediction of the low energy strong coupling constant, or of the m_z/m_w mass ratio). Some conceptual progress may also have a numerical counterpart (example: the solution of the vacuum energy density problem that also allows us to get an estimate of the cosmological constant).

Let us review these various kinds of consequences in the present case of the theory of scale relativity.

Conceptual results

• *Complex nature of wave function*: consequence of nondifferentiability of space-time, that implies a breaking of time reversibility at the level of our elementary description, then a doubling of information, of which complex numbers are the simplest representation [19]. Time reversibility is recovered in terms of a complex process that combines the forward and backward ones. The wave function is the complex action.

• *Probabilistic nature of quantum theory*: consequence of nondifferentiability and fractal nature of space-time, that implies an infinity of geodesics between any couple of events.

• *Correspondence principle*: becomes an equality, thanks to the introduction of complex momentum and energy

• *Schrödinger, Klein-Gordon equations*: demonstrated as equations of geodesics of fractal, nondifferentiable space-time. The quantum terms are implemented from a scale-covariant derivative, and find their origin in a mixing of the effect of the complex representation (consequence of nondifferentiability) and of new second order terms in differential equations (consequence of fractal dimension 2).

• *Quantum / Classical transition*: inherent to the description (since included in the solution to our simplest scale differential equation), identified with the transition from fractal (scale dependence) to nonfractal (scale independence).

• *Divergence of masses and charges*: solved by the new length-scale / mass-scale relation in special scale-relativity; the solution is linked to the new physical meaning of the Planck length-scale.

• *Nature of Planck scale*: becomes a minimal, impassable scale, invariant under dilations, that plays for scale-laws the same role as played by the velocity of light for motion-laws and replaces the zero point as concerns its physical behavior.

• *Nature and quantization of electric charge*: the charge is understood as conservative quantity that comes from the new scale symmetry. Its quantization is a consequence of the limitation on resolutions ratios implied by the new invariant nature of the Planck scale.

• *Origin of mass discretization of elementary particles*: we have suggested that the masses of elementary fermions were of QED origin, and that their discretization was a consequence of charge being quantized.

• *Nature of the cosmological constant*: inverse of the square of a maximal, impassable length-scale $\mathbb{L}$, invariant under dilation, replaces the infinite scale.

• *Vacuum energy density problem*: the energy density is explicitly scale-dependent, so that the Planck energy density does not apply at cosmological scales. The energy density is computed as gravitational self-energy of vacuum fluctuations and is found to vary in terms of resolution as ε^{-6}. Therefore the quantum energy density and the cosmological energy density that manifests itself in terms of cosmological constant become compatible.

• *Large number coincidence*: explained from the above calculation of self-energy density and from the introduction of the maximal invariant length-scale $\mathbb{L}$.

• *Problems of Big-Bang theory* : many problems encountered by the standard Big-Bang theory are automatically resolved in our new framework. The causality problem disappears in terms of Lorentzian dilation laws; there is no need of an inflation phase, then no need to introduce an unknown unobserved arbitrary scalar field to drive it; the age of the universe becomes compatible with that of globular clusters thanks to the introduction of a positive cosmological constant $\Lambda = 1/\mathbb{L}^2$; the problem of the seed of density fluctuations and of the formation and evolution of structures in the universe is resolved in terms of our Schrödinger-like gravitational equation, that yields structures even in uniform density, without any need for initial fluctuations.

Quantified results

• *GUT scale*: becomes in special scale-relativity the Planck mass-scale (that now differs from the Planck length-scale); given by $\log(\lambda_z/\lambda_{GUT}) = \log(\lambda_z/\mathbf{\Lambda}_P)/\sqrt{2} \approx 17/\sqrt{2} \approx 12$.

• *Mass-charge relations*: our interpretation of the charges of fundamental interactions as eigenvalues of the dilation operator acting on resolutions (in other words, as conservative quantities arising from the scale symmetries), of gauge invariance as scale invariance on resolution transformations, and of the 'arbitrary' gauge function as the 'state of scale' $\ln\varepsilon$, leads in special scale-relativity to general mass-charge relations of the form $\alpha\ln(\lambda_c/\mathbf{\Lambda}_P) = k/2$, where k is integer, α is a coupling constant, λ_c a typical Compton scale, inversely related to a mass scale and $\mathbf{\Lambda}_P$ is the Planck length-scale.

• *ElectroWeak scale*: given by the mass / charge relation $\alpha_{0\infty}\ln(\lambda_{EW}/\mathbf{\Lambda}_P) = 1$, i.e., $\lambda_{EW} = \mathbf{\Lambda}_P\, e^{4\pi^2} = 1.397 \times 10^{17}\mathbf{\Lambda}_P \equiv 123$ GeV (while the v.e.v. of the Higgs field is 174 GeV= $123\sqrt{2}$ GeV).

• *Electron scale*: given by the mass / charge relation $\alpha_{0e}\ln(\lambda_e/\mathbf{\Lambda}_P) = 1$, i.e., $m_e = m_P e^{-3/8\alpha_e} \approx 0.5$ MeV.

• *Weak boson mass ratio (value of Weinberg angle)*: we predict that $\alpha_2 = 2\,\alpha_1$ at electroweak scale, so that $m_W/m_Z = \sqrt{10/13}$, and $\sin^2\theta = 3/13$ at this scale.

• *Elementary fermion mass spectrum*: recovered from a cancellation effect between special scale-relativistic corrections and radiative corrections. (However, this is still a model, not a totally constrained theory, because an unknown free parameter remains in this generation mechanism).

• *Top quark mass*: predicted by the above mechanism to fall just beyond the W/Z mass, at 150 ± 50 GeV (observed value: 174 ± 17 GeV).

• *Values of low energy coupling constants*: derived from their renormalization group equations and from the conjecture that the value $1/4\pi^2$ is critical for coupling constants. We find $\alpha_e = 137.08 \pm 0.13$ from $\alpha_{0\infty} = 1/4\pi^2$ and $\alpha_3(m_Z) = 0.1155 \pm 0.0002$ from $\alpha_3(m_{GUT}) = 1/4\pi^2$.

• *Power of galaxy-galaxy correlation function*: the observed value $\gamma = 1.8$ at $\approx 1 - 10$ Mpc is explained as the result of a scale-relativistic correction to the standard value $\gamma = 2$.

• *Structuration of the Solar System*: the observed distribution of mass, angular momentum, eccentricities and positions of planets in the Solar System is accounted for by our 'quantum-gravitational' equation, holding for chaotic system on very large time scales (beyond the horizon of predictibility).

• *Quantization of binary galaxies*: the quantization in terms of $72/n$ km/s observed by Tifft and colleagues in the velocity difference of galaxies in pairs is also predicted by the same approach (Kepler potential).

• *Global redshift quantization of galaxies*: when applied to uniform density, this method predicts a linear quantization (harmonic oscillator) that accounts for the observed 'global' redshift quantization at 36 km/s.

New predictions

• *Precise value of the strong coupling constant*: we predict, as quoted above, $\alpha_3(m_Z) = 0.1155 \pm 0.0002$, more precise than the current value, 0.112 ± 0.003.

• *Precise value of the weak bosons mass ratio*: we predict its exact value $m_W/m_Z = \sqrt{10/13}$ (up to small radiative corrections), while the W mass is presently only poorly known (80.2 ± 0.2 GeV).

• *Breaking of quantum mechanics at high energy*: scale-relativistic 'corrections' will rapidly increase for energies larger than ≈ 100 GeV, since they are no longer cancelled by the appearance of new elementary charged fermions, as happens in the domain 0.5 MeV (electron energy) to 174 GeV (top energy). Provided no new cancellation of electroweak origin takes place above ≈ 100 GeV, we expect the various observed cross sections of particle collisions in future high energy accelerators (LHC...) to depart from their values calculated from standard quantum mechanics (i.e., Galilean scale-relativistic laws). The departure may be expressed, to lowest order, in terms of a scale-varying effective Planck constant [equation (129)].

• *Value of the cosmological constant*: it is predicted to be $\Lambda = 1.36 \times 10^{-56}$ cm^{-2}, under the assumption that the fractal-nonfractal transition for the vacuum energy density occurs at the classical radius of the electron.

• *New planets in the solar system*: some of the 'orbitals' predicted by the theory do not contain observed planets. For some of them this can be understood ($n = 1$ of the inner system is too close to the Sun, $n = 7$ and 10 are destroyed by resonances with Jupiter), but some others may contain objects that have up to now escaped detection ($n = 2$ of the inner system, at 0.185 A.U., $n > 6$ of the outer system).

• *Universal structure of external planetary systems*: we predict that the planetary systems that are expected to be discovered in the near future around nearby stars will be described by the same hydrogen-like orbitals as in our solar system. This prediction seems already to be confirmed by the planetary systems observed around some pulsars [19].

• *Position and velocity structures of stars and stellar associations in our Galaxy*: we predict that the velocity and position distribution of stars in the Galaxy will not be at random, but instead 'quantized' according to our general 'Schrödinger-gravitational' equation. This applies in particular to multiple star systems, to associations and zones of star formation, etc...

• *Structuration of the universe*: in a similar way, galaxies in the universe are predicted by the present theory to form structures at every epochs according to the SU(3) group, that is the symmetry group of the 3-dimensional harmonic oscillator. This is an example of a microscopic-macroscopic connection, SU(3) being, as is well-known, the symmetry group of QCD.

• *Value of power of galaxy correlation function at very large scale*: in our special scale-relativistic theory, the exponent of the galaxy-galaxy correlation function is no longer constant, but varies with scale. While its value is ≈ 1.8 at a scale of ≈ 10 Mpc, we predict that it will fall to ≈ 1.5 at 100 Mpc, then decrease even farther. A precise determination of its variation with resolution would yield a precise measurement of the cosmological constant.

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