A&A manuscript no.
(will be inserted by hand later)

Your thesaurus codes are:
10;02.18.8;02.07.1;07.19.2;07.16.1;08.16.2

ASTRONOMY AND ASTROPHYSICS

Scale-relativity and quantization of exoplanet orbital semi-major axes

L. Nottale[1], **G. Schumacher**[2], **and E.T. Lefèvre**[3]

[1] CNRS, DAEC, Observatoire de Paris-Meudon, F-92195 Meudon Cedex, France
[2] Observatoire de la Côte d'Azur, Département Augustin Fresnel, UMR 6528 du C.N.R.S., av. Copernic, 06130 Grasse, France
[3] 8, rue des Lampes, F-92190 Meudon, France

Astron. Astrophys. 361, 379-387 (2000)
Received 14 April 1999 / Accepted 9 June 2000

Abstract. In a recent study (Nottale 1996b), it was found that the distribution of the semi-major axes of the firstly discovered exoplanets was clustered around quantized values according to the law $a/GM = (n/w_0)^2$, in the same manner and in terms of the same constant $w_0 = 144$ km/s as in our own inner Solar System. The ratio $\alpha_g = w_0/c$ actually stands out as a gravitational coupling constant. The number of exoplanets has now increased fivefold since this first study, including a full system of three planets around Ups And. In the present paper, we apply the same analysis to the new exoplanets and we find that their distribution agrees with this structuration law in a statistically significant way (probability $\approx 10^{-4}$). Such a n^2 law is predicted by the scale-relativity approach to planetary system formation, in which the evolution of planetesimals is described in terms of a generalized Schrödinger equation. In particular, one was able to predict from this model (Nottale 1993) the occurrence of preferential distances of planets at $\approx 0.043\,\mathrm{AU/M_\odot}$ and $\approx 0.17\,\mathrm{AU/M_\odot}$ from their parent stars. The observational data supports this theoretical prediction, since the semimajor axes of $\approx 50\%$ of the presently known exoplanets cluster around these values (51 Peg-type planets).

Key words: relativity – gravitation – solar system – planets and satellites – planetary systems

1. Introduction

Scale relativity, when combined with the laws of gravitation, provides us with a general theory of the structuring of gravitational systems (Nottale 1996a, 1997). In this new approach, we do not any longer follow individual trajectories, but we jump to a statistical description in terms of probability amplitudes. Indeed, we have demonstrated that, under only three simple hypotheses (large number of potential trajectories, fractal geometry of each trajectory and local irreversibility), Newton's equation of dynamics can be transformed and integrated in terms of a generalized Schrödinger equation. This result suggests, in accordance with recent similar conclusions (Ord 1996, Ord & Deakin 1996, El Naschie 1995), that the Schrödinger equation could be universal, i.e. that it may have a larger domain of application than previously thought, but with an interpretation different from that of standard quantum mechanics.

It has been shown (Nottale 1993, 1994), that this approach accounts for several structures observed in the Solar System, including planet distances, eccentricities, and mass distribution (Nottale et al. 1997), obliquities and inclinations of planets and satellites (Nottale 1998a), giant planet satellite distances (Hermann et al. 1998), parabolic comet perihelions (Nottale & Schumacher in preparation). Moreover, it also allows one to predict and understand structures observed on a large range of scales, from binary stars (Nottale & Schumacher 1998), to binary galaxies (Nottale 1996a, Tricottet & Nottale in preparation) and the distribution of galaxies at the scale of the local supercluster (Nottale & Schumacher 1998). A similar kind of approach has been applied by Perdang (1995) to a statistical description of HR diagrams.

It has been also demonstrated that the first newly discovered extra-solar planetary systems come under the same structures, in terms of the same universal constant as in our own Solar System (Nottale 1996b). The system of three planets discovered around the pulsar PSR B1257+12 also agree with the theoretical prediction with a very high precision of some 10^{-4} (Nottale 1996b, 1998b).

The number of exoplanets discovered around solar-like stars has now been multiplied by five since these first studies, so that it seems worthwhile to check again whether the new observational data continue to prove to be non-uniformly distributed.

We indeed find that the semi-major axes of the presently known exoplanet orbits, (measured in terms of the natural gravitational unit for each system, given by the mass of

Send offprint requests to: laurent.nottale@obspm.fr

their parent star), are non-uniformly distributed. Namely, their distribution show peaks of probability density that are consistent with the law $a/GM = n^2/{w_0}^2$, where the constant w_0 is a priori fixed to the value 144 km/s as in our own inner Solar System and in extragalactic data. Moreover, most of these exoplanets (51 Peg-type objects) fall in the fundamental probability density peaks ($n = 1$, $a/M = 0.043$ AU/M$_\odot$) and in the second orbital ($n = 2$, $a/M = 0.17$ AU/M$_\odot$) predicted by the theory.

2. Theory: a short reminder

We have demonstrated (Nottale 1993, 1996a, 1997) that Newton's fundamental equation of dynamics can be integrated in the form of a Schrödinger-like equation under the three following hypotheses:

(i) The test-particles can follow an infinity of potential trajectories: this leads us to use a fluid-like description, $v = v(x(t), t)$.

(ii) The geometry of each trajectory is fractal (of dimension 2). Each elementary displacement is then described in terms of the sum, $dX = dx + d\xi$, of a mean, classical displacement $dx = vdt$ and of a fractal fluctuation $d\xi$ whose behavior satisfies the principle of scale relativity (in its simplest "Galilean" version). It is such that $< d\xi >= 0$ and $< d\xi^2 >= 2\mathcal{D}dt$. The existence of this fluctuation implies introducing new second order terms in the differential equations of motion.

(iii) The motion is assumed to be locally irreversible, i.e., the ($dt \leftrightarrow -dt$) reflection invariance is broken, leading to a two-valuedness of the velocity vector that we represent in terms of a complex velocity, $\mathcal{V} = (v_+ + v_-)/2 - i(v_+ - v_-)/2$.

These three effects can be combined to construct a complex time-derivative operator which writes

$$\frac{\acute{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{V} \cdot \nabla - i\mathcal{D}\,\triangle \tag{1}$$

where the mean velocity $\mathcal{V} = \acute{d}\,x/dt$ is now complex and $\mathcal{D}$ is a parameter characterizing the fractal behavior of trajectories.

Since the mean velocity is complex, the same is true of the Lagrange function, then of the generalized action $\mathcal{S}$. Setting $\psi = e^{i\mathcal{S}/2m\mathcal{D}}$, Newton's equation of dynamics becomes $m\,\acute{d}\,\mathcal{V}/dt = -\nabla\phi$, and can be integrated in terms of a generalized Schrödinger equation (Nottale 1993):

$$\mathcal{D}^2 \triangle \psi + i\mathcal{D}\frac{\partial}{\partial t}\psi = \frac{\phi}{2m}\,\psi. \tag{2}$$

This equation becomes, for a Kepler potential and in the time-independent case:

$$2\mathcal{D}^2 \triangle \psi + (\frac{E}{m} + \frac{GM}{r})\,\psi = 0. \tag{3}$$

Since the imaginary part of this equation is the equation of continuity, $\rho = \psi\psi^\dagger$ can be interpreted as giving the probability density of the particle positions.

Even though it takes this Schrödinger-like form, this equation is still in essence an equation of gravitation, so that it must keep the fundamental properties it owns in Newton's and Einstein's theories. Namely, it must agree with the equivalence principle (Nottale 1996b, Greenberger 1983, Agnese & Festa 1997), i.e., it must be independant of the mass of the test-particle and GM must provide the natural length-unit of the system under consideration. As a consequence, the parameter $\mathcal{D}$ takes the form:

$$\mathcal{D} = \frac{GM}{2w}, \tag{4}$$

where w is a fundamental constant that has the dimension of a velocity.

The solutions of Eq. (3) are given by generalized Laguerre polynomials (see e.g. Nottale et al. 1997). We now assume that such a description can be applied to the distribution of planetesimals in the protoplanetary nebula. We expect them to fill these "orbitals", then to form a planet by accretion as in the standard models of planetary formation. But the new point here is that only some particular orbitals are allowed, so that the semi-major axes of the orbits of the resulting planets are quantized according to the law:

$$a_n = \frac{GMn^2}{w^2}, \tag{5}$$

where n is an integer. In an equivalent way, using Kepler's third law that relates the semimajor axis a to the orbital period P, $(a/GM)^3 = (P/2\pi GM)^2$, the average velocity of the planet, $v = 2\pi a/P = (GM/a)^{1/2}$, is expected to have a distribution peaked at $v_n = w/n$, i.e. in Solar System units (AU, year, M$_\odot$ and Earth velocity):

$$(a/M)^{1/2} = (P/M)^{1/3} = \frac{1}{v} = \frac{n}{w}. \tag{6}$$

3. On the nature of the fundamental ratio $\alpha_g = w/c$: a gravitational coupling constant

It may be useful at that step to be more specific about the nature of the new fundamental constant, even though a more detailed analysis will be given in a forthcoming work (Nottale in preparation). The meaning of w/c can be anticipated from a comparison with the quantum hydrogen atom. Indeed, it is well known that, on its fundamental level, the average orbital velocity of an electron is given by $< v > /c = \alpha$, where α is the fine structure constant, i.e. the coupling constant of electromagnetism.

In the macroscopic case considered here the problem is purely gravitational, but w still gives the average velocity of the fundamental level. Let us demonstrate that w/c plays the role of a gravitational coupling constant. The fine structure constant appears in the expression of the Coulomb force when the square of the electric charge is expressed in terms of quantum units, i.e.:

$$F_{em} = \frac{e^2}{4\pi\epsilon_0 r^2} = \alpha\,\frac{\hbar c}{r^2}. \tag{7}$$

Now the correspondence between the standard microscopic quantum theory and the macroscopic quasi-quantum situation described here is given by $\hbar \to 2m\mathcal{D}$, so that we are lead to write:

$$F_g = \alpha_g \, \frac{2m\mathcal{D}\, c}{r^2}. \tag{8}$$

Now, since $\mathcal{D} = GM/2w$ and $F_g = GMm/r^2$, the identification of both expressions of the force implies:

$$\alpha_g = \frac{w}{c}. \tag{9}$$

This establishes w/c as a macroscopic gravitational coupling constant, in agreement with Agnese & Festa (1997), and Agop et al. (1999).

In Nottale 1996b, the value of w was determined to be $w = 144.7 \pm 0.5$ km/s from several different quantization effects ranging from the scale of the solar system to extragalactic scales, in agreement with Tifft & Cocke (1984) own precise determination (144.9 km/s) from the Tifft effect of redshift quantization. This value corresponds to an inverse coupling constant $\alpha_g^{-1} = 2072 \pm 7$.

The question of a theoretical prediction of the value of this constant might reveal to be a difficult one, owing to the fact that there is still no theoretical understanding, in the standard model of elementary particles, of the value of the electromagnetic coupling constant itself (however, see Nottale 1996a). A full discussion of this problem will be considered elsewhere.

However, one can already remark here that its solution is expected to involve connections between local and global scales, i.e. it might be related with Mach's principle. Recall that, in other contributions , new dilation laws having a log-Lorentz form have been introduced (Nottale 1992), that lead to re-interpret the length-scale of the cosmological constant $\mathbb{L}_U = \Lambda^{-1/2}$ and the Planck length-scale $\mathbb{L}_{Pl}$ as impassable, respectively maximal and minimal length-scales, invariant under dilations of resolutions (see e.g. Nottale 1993, 1996a): i.e., they would play for scale transformations of resolutions a role similar to that of the velocity of light for motion transformations.

Their ratio defines a fundamental pure number, $\mathbb{K} = \mathbb{L}_U/\mathbb{L}_{Pl}$. The logarithm of this ratio has been found to have the numerical value $\mathbb{C}_U = \ln \mathbb{K} = 139.83 \pm 0.01$, i.e. $\mathbb{K} = 5.3 \times 10^{60}$ from an analysis of the vacuum energy density problem (Nottale, 1993, 1996a). This value corresponds to $\Omega_\Lambda = 0.36h^{-2}$(= 0.7 for a Hubble constant $H_0 = 70$ km/s.Mpc) and it has been corroborated by recent indirect measurements of the cosmological constant using SNe I (Garnavich et al. 1998; Perlmutter et al. 1998; Riess et al. 1998).

Moreover, one is also lead, in the scale-relativistic framework, to give a new interpretation of gauge invariance as being invariance in the resolution space. The universal limit on possible scale ratios thus implies a quantization of coupling constants (this amounts to defining a wave in scale space). This allows one to set new fundamental relations between coupling constants and Compton lengths over Planck length ratios, that typically write (Nottale 1996a):

$$\alpha \, \ln(\frac{\lambda_c}{\mathbb{L}_{Pl}}) = \mathrm{cst}. \tag{10}$$

When it is applied to the electron structure, which is upper limited in scale by its Compton length and lower limited by the Planck length-scale, this method yields a relation between the electromagnetic coupling as it is defined in the electroweak theory, $\frac{8}{3}\alpha$, and the electron mass in Planck mass unit:

$$\frac{8}{3}\alpha \, \ln(\frac{m_{Pl}}{m_e}) = 1. \tag{11}$$

This relation is satisfied within 0.3 % by the experimental values of the fine structure constant and of the electron mass. Recall that this method also allows one to suggest a solution to the hierarchy problem between the GUT and electroweak scale (WZ). Indeed we have suggested, in the minimal standard model reformulated in the special scale-relativity framework, that bare couplings are given by the critical value $1/4\pi^2$, so that one can define a fundamental scale given by

$$\ln(\frac{m_{Pl}}{m_{WZ}}) = 4\pi^2, \tag{12}$$

which is nothing but the electroweak scale (≈ 90 GeV).

We can now apply the same reasoning to gravitation in the new framework. Indeed, contrarily to what happens in the classical theory, the equation of motion (Eq. 2) can be shown to be gauge invariant. If the potential ϕ is replaced by $\phi + GMm\, \partial\chi(t)/c\partial t$, where the factor GMm ensures a correct dimensionality, then Eq. (2) remains invariant provided ψ is replaced by $\psi\, e^{-i\alpha_g\chi}$, with α_g related to $\mathcal{D}$ by:

$$\alpha_g \times 2m\mathcal{D} = \frac{GMm}{c}, \tag{13}$$

which is the previously established relation for $\alpha_g = w/c$. Therefore, gauge invariance allows one to demonstrate the form of the coefficient $\mathcal{D}$ that we obtained from dimensional considerations. The advantage of this result is that it will be generalizable to gravitational potentials different from the Kepler one.

Finally, similarly to the electromagnetic case, we can interpret the arbitrary gauge function χ, up to some numerical constant, as the logarithm of a scale factor $\ln\rho$ in resolution space. In the special scale-relativity framework, such a scale factor is limited by the ratio of the maximal cosmic scale over the Planck scale, i.e. $\ln\rho < \mathbb{C}_U$. This limitation of χ in the phase of the wave function ψ implies a quantization of its conjugate quantity α_g, following the relation:

$$k\, \alpha_g\, \mathbb{C}_U = 1. \tag{14}$$

The numerical constant k remains to be determined. A possible suggestion is that $k = \frac{3}{2}\pi^2$, which yields a predicted value of $\alpha_g^{-1} = 2070.10 \pm 0.15$ and $w_0 = 144.82 \pm$

0.01 km/s, in good agreement with it precise observational determinations. Reversely, from such a relation, if it was confirmed, a precise measurement of w would provide one with a new way of determining the cosmological constant.

4. Comparison with observational data and statistical analysis

Table 1 gives the periods P of the newly observed exoplanets. The masses M of the parent stars are taken from the compilation of Marcy et al. (1999). Only planetary companions are considered here, to the exclusion of brown dwarfs (but see also Sect. 6), using the criterion that their mass be smaller than 13 Jupiter masses (see e.g. Schneider 2000).

The references for the observed orbital periods and for the star masses are as follows: (1) Lang (1992); (2) Mayor & Queloz (1995); (3) Butler et al. (1998); (4) Butler et al.(1997); (5) Noyes et al. (1997); (6) Fischer et al. (1998); (7) Marcy et al. (1999); (8) Cochran et al. (1997); (9) Marcy et al. (1998); (10) Butler & Marcy (1996); (11) Queloz et al. (2000); (12) Marcy & Butler (1996); (13) Mazeh et al. (1996); (14) Mayor et al. (1999); (15) Butler et al. (1999); (16) Santos et al. (2000); (17) Udry et al. (2000); (18) Mayor et al. (1998), Marcy (2000); (19) Charbonneau et al. (1999), Henry et al. (2000); (20) Vogt et al. (2000); (21) Kurster et al. (1998); (22) Marcy (2000).

We define an effective "quantum number"

$$\tilde{n} = 4.83\,(P/M)^{1/3} \tag{15}$$

which is computed directly from the observational data, i.e., the values of the orbital period P and of the star mass M for each planet and its parent star. The number $4.83 = 144/29.79$ expresses, in terms of Solar System units (AU and $M_\odot$), the value $w_0 = 144$ km/s which characterizes galactic and extragalactic systems (Tifft 1977) and also our own inner Solar System (Nottale 1996b, Nottale et al 1997). Indeed the average Earth velocity is 29.79 km/s.

Therefore, our theoretical prediction can be summarized by the statement that the distribution of the values of $\tilde{n} = 144/v$ must cluster around integer numbers. These values are given in column 5 of Table 1 and their distribution is plotted in Fig. 1.

Note that, since the main source of error is the uncertainty on the star mass (usually $\approx 10\%$), the relative uncertainty on $\tilde{n}$ is $\approx 0.03\,\%$. Then our theoretical prediction according to which $\tilde{n}$ must be close to an integer becomes more difficult to be checked beyond $n > 6$, since the error bar becomes too large ($> 20\%$).

The values of the masses of the parent stars have been taken from the compilation of Marcy et al. (1999), and for more recently discovered planets, from Mayor et al. (1999: HD 75289), Charboneau et al (1999: HD 209458), Vogt et al (2000), Marcy (2000). For two of these planets, HD 177830 and HD 10697, the masses being badly determined and in contradiction with the mass expected from their spectral type, we have taken an average value (1.0 solar mass).

We have plotted in Fig. 2 the histogram of the differences δn between $\tilde{n}$ and the nearest integer, for the data of Table 1. As can be checked in this figure, as well as in Table 1 and Fig. 1, we verify that the observed values of $\tilde{n}$ indeed cluster around integer values.

Let us make a statistical analysis of this result. We recall that we have performed no fit of the data. Indeed, we look for clustering around integer values of the ratio $144/v$, where the value $w_0 = 144$ km/s is taken from independent results (e.g., extragalactic data on binary galaxies) and where v is calculated from the observed star mass and planet orbital period. As a consequence the zero hypothesis corresponds to a uniform distribution of δn values.

A Kolmogorov-Smirnov one-sample test yields a maximum difference $D = 0.31$ between the observed cumulative distribution and that of a uniform distribution. For n=42 points, this result has a probability $p < 10^{-3}$ to be obtained by chance.

We can also make an independent test by separating the δn domain into two equal intervals respectively of high probability, [-0.25,+0.25] and low probability, [0.25,0.75] . It can be seen in Fig. 2 that among 42 points, 33 fall in the interval [-0.25,+0.25] and only 9 in [0.25,0.75]. The probability that, for 42 trials, 9 events (or less) fall in a 1/2 large interval and 33 in the complementary one is $p = \sum_{i=0}^{i=9} (42, i) \times 2^{-42} = 1.3 \times 10^{-4}$, where $(42, i)$ denotes a binomial coefficient. Therefore we can exclude at better than the 3σ level of statistical significance that such a result be obtained by chance.

Finally, we have performed the same analysis by taking star masses deduced from their spectral type (from Allen 1973). One finds essentially the same statistical result. Among 42 points, 33 fall in the interval [-0.25,+0.25] and 9 in [0.25,0.75].

5. The triple system around Upsilon Andromedae

After the submission of the present paper, the evidence that Ups And has two additional planets was reported (Butler et al 1999). This report provides us with a new occasion to test for the n^2 law in a relative way using a three planet system, as already done for the three planets around the pulsar PSR B1257+12 (Nottale 1996b, 1998b). As can be seen in Fig. 3, the result obtained is very precise. Indeed, we find that the observed periods of the three planets Ups And b, c, d follow the above relation:

$$\frac{P_d^{1/3} - P_c^{1/3}}{P_c^{1/3} - P_b^{1/3}} = 1.006. \tag{16}$$

This is a strong indication that $n_d - n_c = n_c - n_b$, which can be checked directly from the "absolute" analysis (see Table 1) according to which the three planets rank respectively n =1, 4, 7 with a precision better than 10%.

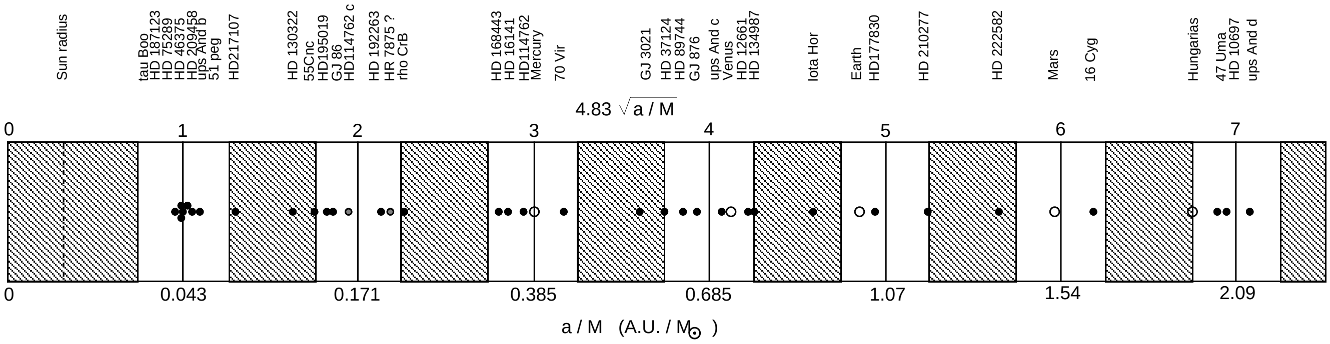

Fig. 1. Observed distribution of $\tilde{n} = 4.83\,(P/M)^{1/3} = 144/v$ where the orbital period P and the star mass M are taken in Solar System units (AU and $M_\odot$), for the recently discovered exoplanet candidates (black dots) and for the planets of our inner Solar System (white dots). The planet around HR 7875, that remains unconfirmed, and the second planet around HD 114762, the existence of which is tentatively suggested here, are plotted as grey dots. The grey zone stands for the theoretically predicted low probability of presence of planets and the white zones for high probability. The error bars are typically of the order of $0.03\,\tilde{n}$.

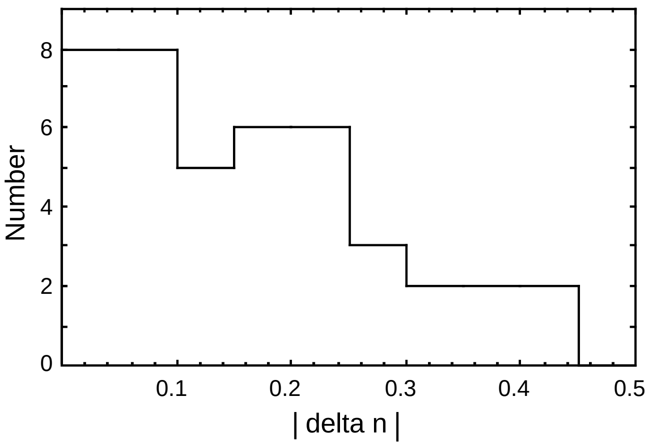

Fig. 2. Histogram of the values of $|\delta n| = |144/v - \tilde{n}|$ for planets in the inner solar system and exoplanets. The mean velocity is computed from the original data using Kepler third law as $v = (M/P)^{1/3}$, where M is the parent star mass and P is the planet period (in solar system units). Under the standard, no struturation, hypothesis, the distribution should be uniform in the interval [0,0.5] . On the contrary, it is found that 33 objects among 42 fall in the first half interval and only 9 in the second. The probability to obtain such a result by chance (which is as getting 9 heads while tossing a coin 42 times) is $\approx 10^{-4}$.

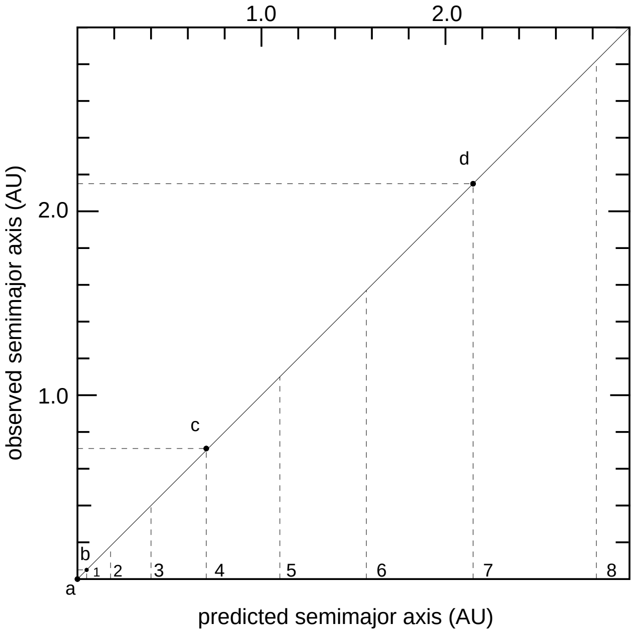

Fig. 3. The system of three planets around Ups And. The observed values of the semi-major axes of the three planet orbits are compared with the possible values predicted for the peaks of probability density, $a/GM = (n/w_0)^2$ with $w_0 = 144$ km/s. An excellent agreement is found for $n = 1, 4, 7$. Note that the large eccentricities of the orbits of planets c and d may explain the emptiness of the other "orbitals".

Let us compute the probability to get such an agreement of the observed orbital periods with the theoretical prediction for this system. The differences between the observed values of $\tilde{n}$ and the nearest integers for the three planets are respectively 0.09(b), 0.08(c) and 0.09(d). The probability to get such a result by chance is only $p \approx (0.2)^3 < 10^{-2}$. This is confirmed by a Kolmogorov-Smirnov test: we obtain $D = 0.82$ for 3 points, which corresponds to a 1% probability.

6. The planet around HD 209458 and other 51 Peg-like planets

More recently, a 51 Peg-like planet, initially found from a precision Doppler survey, has been detected in a more direct way by its transit across its parent star, HD 209458 (Charbonneau et al. 1999, Henry et al. 2000, Robichon & Arenou 2000). This photometric detection allows one

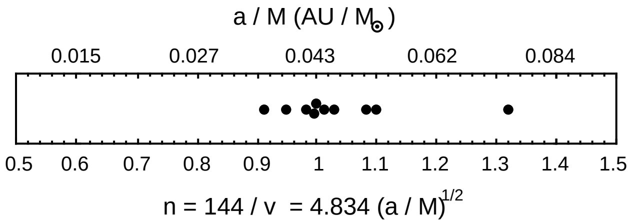

Fig. 4. Observed distribution of $\sqrt{a/M}$ for exoplanets and brown dwarfs with $a/M < 0.1$, where a is the orbital semi-major axis and M the parent star mass, compared with the theoretical prediction (peak of probability density of the orbital $n = 1$ expected at $a/M = 0.043\,\mathrm{AU/M_\odot}$).

to determine the inclination of the orbit, then the exact mass ($0.63\,\mathrm{M_J}$) and to confirm definitely the existence of a Jupiter-like planet. Therefore this planet deserves a particular study. From Charbonneau et al. (1999) data, $M = 1.1\,M_\odot$ and $P = 3.5245(3)$ days, one obtains $\tilde{n} = 144/v = 0.997$, while Henry et al. 2000) data, $M = 1.03\,M_\odot$ and $P = 3.524(5)$ days, give $\tilde{n} = 144/v = 1.018$. This planet is then found to lie in a precise way at the predicted peak of probability density of the fundamental "orbital" $n = 1$. This argument is reinforced by the recent probable spectral detection (Cameron et al. 1999), from Doppler-shifted reflected starlight, of the planet orbiting Tau Boo, which is also a fair $n = 1$ planet ($\tilde{n} = 144/v = 0.95$).

Eight 51 Peg like planets with $a/M < 0.1\,\mathrm{AU/M_\odot}$ have now been discovered, and also two brown dwarfs (Mayor et al. 1997), HD 98230 ($\tilde{n} = 1.02$) and HD 283750 ($\tilde{n} = 0.91$). Except for one object at $\tilde{n} = 1.32$ (HD 217107), they are all clustered between $\tilde{n} \approx 0.9$ and 1.1 (see Fig. 4). The mean orbital velocity of these planets is $< v >= 143 \pm 3$ km/s, i.e., they constitute a direct and precise achievement of the fundamental constant w_0.

Before concluding, note that the observed distribution of small $\tilde{n}$ values suggests that there exists a new hierarchy level of structuration based on $w = 432$ km/s for 51 Peg types exoplanets (which would imply the possibility of finding exoplanets lying at $a/M = 0.019\,\mathrm{AU/M_\odot}$ from their star). This would agree with our result (Nottale, Schumacher & Gay 1997) that the $n = 1$ orbital of the outer solar system is actually sub-structured in terms of the inner solar system itself. Moreover, the distribution of the perihelions of intramercurial comets show peaks at predicted positions corresponding to constants $w = 432 = 3 \times 144$ km/s and $1296 = 3 \times 432$ km/s (Nottale & Schumacher in preparation).

7. Discussion

7.1. Theory

Concerning the theory, we shall briefly discuss three points. A more complete discussion will be published elsewhere (Nottale in preparation).

(i) Let us first compare our approach with the standard theory of gravitational structure formation and evolution. We write instead of the Euler-Newton equation and of the continuity equation a unique, complex, generalized Euler-Newton equation that can be integrated in terms of a Schrödinger equation, completed by the Poisson equation. The square of the modulus of the Schrödinger equation then gives the probability density ρ. Now, when the 'orbitals', which are solutions of the motion equation, can be considered as filled with the particles (e.g., planetesimals in the case of planetary systems formation), their mass density is proportional to the probability density. By separating the real and imaginary parts of the Schrödinger equation we get respectively a generalized Euler-Newton equation and the continuity equation (which is therefore now part of the dynamics), so that our system becomes:

$$m\,(\frac{\partial}{\partial t} + V \cdot \nabla)V = -\nabla(\phi + Q), \tag{17}$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho V) = 0, \tag{18}$$

$$\Delta\phi = -4\pi G\rho. \tag{19}$$

Then this system of equations is equivalent to the classical one, except for the introduction of an extra potential term Q that writes:

$$Q = -2m\mathcal{D}^2 \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}. \tag{20}$$

In this case the new potential term (the "Bohm potential") is a function of the density of matter, as the usual Newton potential.

We recover the standard theory in the limit $\mathcal{D} \to 0$. As a consequence, from the viewpoint of the application of the present approach to the formation and evolution of gravitational systems, the main question becomes to demonstrate from observational data that the parameter $\mathcal{D}$ does exist and is non vanishing, then to determine its form and its values in the various physical configurations. This is one of the goal of the present paper and of forthcoming works.

Conversely, the existence of this additional term demonstrates that there is indeed new physics here, i.e. that one cannot obtain such a system of equations from a simple extrapolation of the standard approach.

(ii) This remark leads us to briefly discuss other related approaches which also suggested the use of a quantum-like description for planetary systems and for other gravitational structures.

The suggestion to use the formalism of quantum mechanics for the treatment of macroscopic problems, in particular for understanding structures in the solar system, dates back to the beginnings of the quantum theory (see e.g. Jehle 1938, Liebowitz 1944, and Reinisch 1998 for additional references). However these early attempts suffered from the lack of a convincing justification of the use of a quantum-like formalism and were hardly generalizable (e.g., the Schrödinger-like equation derived by Liebowitz applied only to stationary states of n particles and allowed only for a real (non-complex) wave function). Anyway these works clearly anticipated the recent understanding of the universality of the Schrödinger equation.

More recently, several works attempted to develop a macroscopic quantum theory under the motivation of describing the various Tifft effects of redshift quantization (Greenberger 1983, Dersarkissian 1984, Carvalho 1985, Cocke 1985). The problem with these attempts is that they did not allow an understanding of the origin of the observed quantization and of its meaning. Recall indeed that our interpretation of these effects is different from theirs. They were interpreted by Tifft as an occurrence of anomalous, precisely quantized non-Doppler redshifts, while in the present approach we interpret them as peaks of the probability distribution of standard velocity redshifts. The present result that the solar system planets and exoplanets (for which we are certain that we deal with true velocities) show the same structures supports our interpretation.

A third, better theoretically motivated, approach, has been the suggestion to use Nelson's stochastic mechanics (Nelson 1966) as a description of the diffusion process in the protosolar nebula (Albeverio et al. 1983, Blanchard 1984). The problem with such a suggestion is that Nelson's twin diffusion process is not a standard diffusion process. Indeed, in standard diffusion one can establish a forward Kolmogorov equation (also called the Fokker-Planck equation), then a backward Kolmogorov equation in which the average velocity is the same as in the forward one (see e.g. Welsh 1970). On the contrary, Nelson's theory introduces, in addition to the forward Kolmogorov equation (which he calls forward Fokker-Planck equation), a "backward Fokker-Planck equation" in which the average backward velocity is different from the forward one. This backward Fokker-Planck equation is therefore incompatible with the backward Kolmogorov equation of standard diffusion. Moreover, the form of the mean acceleration in stochastics mechanics must be arbitrarily postulated. Fianally, a justification is lacking for the application of such a non standard diffusion process to macroscopic systems.

Recall that in our own present attempt, these problems are overcome since (i) the two-valuedness of the mean velocity vector is explained as a consequence of the $dt \to -dt$ symmetry breaking (local irreversibility); (ii) we use neither Fokker-Planck equations nor a diffusion description; (iii) our basic equation, written in terms of the new "covariant" time derivative operator, keeps the standard form of the equation of dynamics, and it now includes the continuity equation.

(iii) A last point that we shall briefly discuss here is Reinisch (1998)'s proposal according to which the $n = 1$ mode should correspond to purely radial motion and then be singular and therefore forbidden. He suggests to use non linear modes that give a distance law $a/a_B = n[(n-1) + \sqrt{(n-1)/2}]$. These solutions, when fitted to the inner solar system, yield $a_B = 0.0422$ AU, and then give probability peaks at 0.14, 0.38, 0.71, 1.14 and 1.67 AU. The agreement is bad for the Earth (1 UA) and Mars (1.52 UA), but, more importantly, the theoretical prediction of this model for extrasolar planets at intramercurial distances is one unique peak at $a/M = 0.14\,\mathrm{AU/M_\odot}$. This is clearly rejected by the observed distribution of exoplanets close to their stars (see Table 1 and Figs. 1 and 4), in particular by the well-defined peak at $0.043\,\mathrm{AU/M_\odot}$. The problem raised by Reinisch is that the fundamental mode is characterized by a secondary quantum number $l = n - 1 = 0$ for $n = 1$, and should therefore correspond to zero angular momentum. However, one should not forget that this behaviour is obtained from a highly simplified treatment of planetary formation, namely, pure Kepler two-body problem with spherical symmetry. This is clearly an oversimplication, since a more complete approach should also account for flattening, self-gravitation of the disk and other effects. The work aiming at developing such a more realistic treatment is now in progress. Anyway, the above question is already simply solved by jumping to a 2-D Schrödinger equation, in order to describe the high flattening of the initial dust disk: we indeed find in this case that the radial modes are suppressed, since the secondary quantum number now varies from 1/2 to $n-1/2$ (Nottale in preparation). Moreover, in all cases studied up to now, the $n = 1$ orbital happens to be substructured.

7.2. Error analysis

The number of exoplanets (≈ 35) now known allows one to perform an analysis of the distribution of errors. As already remarked, the main source of error is about the parent star mass. The relative error is $\approx \pm 10\%$, so that the uncertainty on $\tilde{n}$, which is proportional to $M^{-1/3}$, is $\approx 0.033\,n$.

Then we can compute the expected number of exoplanets for which δn (the difference between $\tilde{n} = 4.83\,(P/M)^{1/3}$ and the nearest integer) fall outside the interval [-0.25, +025] due to the error on mass alone. By using a Student law corresponding to the observed number of exoplanets in each "orbital", we find a total number of 4.1 exoplanets expected to be discrepant, to be compared with the observed number of 9 among 35 exoplanets.

This indicates that, as expected from our model, there is an intrinsic dispersion around the "quantized" values

$(n/w_0)^2$ of the semi-major axes. This dispersion is expected to be non zero, but smaller than that of the calculated orbitals, given by generalized Laguerre polynomials. Indeed, we interpret these orbitals as describing the density distribution of *planetesimals* in the initial disk. The n^2 values give their density peaks, so that, after the accretion process, the planet is expected to lie with higher probability at about this distance from its star. A theoretical estimate of the expected dispersion of the planet positions around the peak values remains to be done. It should depend on the various conditions and perturbations that intervene during the accretion process, as indicated, e.g., by the fact that the observed dispersion for the three planets around the pulsar PSR B1257+12 (Nottale 1996b) is far smaller that for the planets of our Solar System.

The inner Solar System, for which the planet positions are known with precision, and the exoplanet $n = 1$ orbital, for which the contribution of the error on the star mass remains small, allow us to obtain an observational estimate of the intrinsic dispersion on $\tilde{n}$. We get for the inner Solar System $\sigma = 0.121$ (7 objects), a value which is confirmed by the $n = 1$ exoplanets (7 objects), for which $\sigma = 0.124$. We can now make again the preceding analysis by combining the intrinsic dispersion, assumed to be $\sigma = 0.125$, and the error from the mass star (see Table 2). We find a total expected number of ≈ 8 discrepant exoplanets (outside the interval [-0.25,0.25]), which compares well with the observed number of 9.

7.3. Consequences for exoplanets

We have shown that the new exoplanets discovered since three years agree in a statistically significant way with the theoretical prediction of the scale-relativity approach. This is a confirmation of the result which was established with the firstly discovered ones (51 Peg, ups And, tau Boo, 55 Cnc, HD114762, 47 UMa, see Nottale 1996b). The existence of a candidate planet around Prox Cen, which was taken into account in (Nottale 1996b), has not been confirmed since and therefore it was excluded from the present analysis. The candidate planet around HR 7875 also remains to be confirmed.

One of the possible consequences of our result is that the quantization of planet interdistances to their stars may be used as a filter to help discovering secondary planets. For example, in the case of HD 114762, the power spectrum published by Mazeh et al. (1996) shows, in addition to the main peak at 84 days, a secondary smaller but isolated peak corresponding to a period of 22.2 days. Such a peak in a power spectrum, though statistically unsignificant when there is no theoretical prediction, may become significant provided its value was a priori predicted. In the case of HD 114762, the secondary small peak is one of the possible periods for this star predicted from the n^2 law. Indeed, we find $a/M = 0.154$ AU/M$_\odot$ and $\tilde{n} = 1.90$. If confirmed with future improved data, this would yield a new object on the $n = 2$ orbital (see Fig. 1), in addition to the already discovered $n = 3$ planet around this star.

Note also that our theoretical methods may help solving some of the problems encountered in the standard models of planetary formation. In the description of the protoplanetary nebula based on the present approach, the 51 Peg-type planets could be formed in situ, precisely at the distances close to their stars where they are observed. But our generalized Schrödinger equation could also be applied as a statistical description of the chaotic motion of a planet formed at a Jupiter distance, and which would spiral in the disk toward the inner planetary system. In both cases, the same final position is predicted, in the first case as a peak of planetesimal density and in the second as a peak of probability of presence for the planet.

Anyway, it has already been remarked (e.g. Marcy et al. 1999) that in the standard paradigm of planetary formation, the distribution of semi-major axes and eccentricities of giant planets presents an unsolved puzzle. Indeed, in the inward migration scenario, circular orbits are expected while most exoplanets have non-circular orbits, and, moreover, no mechanism is known to halt the migration. Our present result according to which, when plotted in terms of a/M instead of a, this migration should be stopped in most cases at ≈ 0.17 AU/M$_\odot$, and ≈ 0.043 AU/M$_\odot$ is therefore still harder to understand in the standard scheme, while these values are the theoretically predicted ones in our own model.

We conclude this discussion by noting that, in a recent paper, Laskar (2000) developed a simplified model of planetary accretion that also yields a n^2-like distance law (of the form $\sqrt{a} = \mathrm{b}n + \mathrm{c}$) for a particular choice of the initial mass distribution. This is an interesting convergence of the standard approach with the generalized one considered here; however, the standard methods will be confronted to the difficult problems of explaining why, as well in the Solar system as for extrasolar systems, the constant c is zero, the ratio $\mathrm{b}/\sqrt{M}$ is a universal constant, and why this n^2 law holds even in the case of one single planet.

8. Conclusion

We conclude that the distribution of the semi-major axes of extrasolar planets around solar-like stars and of planets in our own inner Solar System, is consistent with a clustering around quantized values given by $a/GM = (n/w_0)^2$, where $w_0 = 144$ km/s, and n is integer.

Recall that comparing the planetary systems one with another in terms of a/M is required already in Newton's theory of gravitation (and in Einstein's theory), since the star mass gives the natural length unit of the whole system, as can be seen e.g. in the expression of Kepler's third law or of Newton's law of gravitation.

Independently of our theoretical approach, one can therefore verify in a purely empirical way that the exoplanet semi-major axes are distributed, once related to their star mass, in the same way as in our inner Solar System, so that, in opposition to the standard claim, we conclude that all exoplanetary systems discovered so far are similar to our own system.

We finally recall that the prediction that planets should be found around $a/M = 0.043$ AU/M$_\odot$ and 0.17 AU/M$_\odot$ was made in a "blind" way several years before the discovery of the new exoplanets (Nottale 1993, 1994), and could have been used to anticipate the possibility of this discovery.

Acknowledgments. We gratefully thank Drs. J. Gay, J. Perdang, J. Schneider and M. Tricottet for helpful discussions.

References

Agnese A.G., Festa R., 1997, Phys. Lett. A227, 165
Agop M., Matsuzawa H., Oprea I.et al., 1999, Australian. J. Phys. 53, 217
Albeverio S., Blanchard Ph., Hoegh-Krohn R., 1983, Exp. Math. 4, 365
Allen C.W., 1973, Astrophysical Quantities (The Athlone Press, London)
Blanchard Ph., 1984, Acta Physica Austriaca Suppl. XXVI 185
Butler P., Marcy G., 1996, ApJ 464, L153
Butler P., Marcy G., Williams E., Hausert H., Shirts P., 1997, ApJ 474, L115
Butler P., Marcy G., Vogt S., Apps K., 1998, PASP 110, 1389
Butler P., Marcy G., Fisher D. et al., 1999, ApJ 526, 916
Cameron A.C., Horne K., Penny A., James D., 1999, Nat 402, 751
Carvalho J.C., 1985, Lett. Nuovo Cimento 44, 137
Charbonneau D., Brown T.M., Latham D.W., Mayor M., 1999, ApJ 529, 45
Cocke W.J., 1985, ApJ 288, 22
Cochran W., Hatzes A., Butler P., Marcy G., 1997, ApJ 483, 457
Dersarkissian M., 1984, Lett. Nuovo Cimento 40, 390
El Naschie M.S., 1995, in Quantum mechanics, Diffusion and Chaotic Fractals, eds. M.S. El Naschie, O.E. Rossler and I. Prigogine, (Pergamon), pp. 93, 185
Fischer D., Marcy G., Butler P., Vogt S., Apps K., 1998, PASP 111, 50
Garnavich P.M., Jha S., Challis P. et al., 1998, ApJ 509, 74
Greenberger D.M., 1983, Found. Phys. 13, 903
Henry G., Marcy G., Butler P., Vogt S., 2000, ApJ 529, L41
Hermann R., Schumacher G., Guyard R. , 1998, A&A 335, 281
Jehle H., 1938, Zeitschrift fur Astrophysics 15, 182
Kurster M., Hatzes A., Cochran W. et al., 1998, The ESO radial velocity planet search program, in Proceeding Workshop Science with Gemin, 7-10 December 1997, Florianopolis, Brazil, B. Barbuy (ed.)
Lang K.R., 1992, Astrophysical Data. Planets and Stars (Springer)
Laskar J., 2000, Phys. Rev. Lett. 84, 3240
Liebowitz, B., 1944, Phys. Rev. 66, 343
Marcy G. W., 2000, http://astron.berkeley.edu/gmarcy/
Marcy G. W., Butler R. P., 1996, ApJ 464, L147
Marcy G., Butler P., Vogt S., Fischer D., Lissauer J., 1998, ApJ 505, L147
Marcy G., Butler P., Vogt S., Fischer D., Liu M., 1999, ApJ 520, 239
Mayor M., Queloz D., 1995, Nat 378, 355
Mayor M., Queloz M., Udry S., Halbwachs, 1997, From brown dwarfs to planets, in Bioastronomy 96 , Capri, Italy, July 1 - 5 1996 (IAU Colloquium 161)
Mayor M., Queloz D., Beuzit J.-L. et al., 1998, Protostars and Planets IV (V. Manings Ed.)
Mayor M., Naef D., Udry S. et al., 1999, Exoplanet message no. 116, 2 Feb. 1999
Mazeh T., Latham D. W., Stefanik R. P., 1996, ApJ 466, 415
Nelson E., 1966, Phys. Rev. 150, 1079
Nottale L., 1992, Int. J. Mod. Phys. A7, 4899
Nottale L., 1993, Fractal Space-Time and Microphysics: Towards a Theory of Scale Relativity (World Scientific)
Nottale L., 1994, in Chaos and Diffusion in Hamiltonian Systems, D. Benest & C. Froeschlé eds, (Frontières), p. 173
Nottale L., 1996a, Chaos, Solitons & Fractals 7, 877
Nottale L., 1996b, A&A Lett. 315, L9
Nottale L., 1997, A&A 327, 867
Nottale L., 1998a, Chaos, Solitons & Fractals 9, 1035
Nottale L., 1998b, Chaos, Solitons & Fractals 9, 1043
Nottale L., Schumacher G., 1998, in Fractals and Beyond: complexities in the sciences, ed. M.M. Novak (World Scientific), p. 149
Nottale L., Schumacher G., Gay J., 1997, A&A 322, 1018
Noyes R., Jha S., Korzennik S. et al., 1997, ApJ 483, L111
Ord G.N., 1996, Chaos, Solitons & Fractals 7, 821
Ord G.N., Deakin A.S., 1996, Phys. Rev. A 54, 3772
Perlmutter S., Aldering G., Della Valle M. et al., 1998, Nat 391, 51
Perdang J., 1995, in Chaos and Stellar Evolution, Proceedings of the 32th Liege International Colloquium, eds. A. Noel, D. Fraipont-Caro, M. Gabriel, N. Grevesse, P. Demarque (Derouaux, Liege), p. 147
Queloz D., Mayor M., Weber L. et al., 2000, A&A 354, 99
Reinisch G., 1998, A&A 337, 299
Riess A.G., Filippenko A.V., Challis P. et al., 1998, AJ 116, 1009
Robichon N., Arenou F., 2000, A&A 355, 295
Santos N., Mayor M., Naef D. et al., 2000, A&A 356, 599
Schneider J., 2000, http://www.obspm.fr/encycl/catalog.html
Tifft W.G., 1977, ApJ 211, 31
Tifft W.G., Cocke W.J., 1984, ApJ 287, 492
Udry S., et al., 2000, A&A 356, 590
Vogt S., Marcy G., Butler P., Apps K., 2000, ApJ in press
Welsh D.J.A., 1970, in Mathematics applied to Physics (Springer), p. 465

Star Name	Star mass	P	a/M	$\tilde{n}$	Ref.
Sun/Merc.	1.00 ± 0.00	0.241	0.387	3.01	(1)
Sun/Venus	1.00 ± 0.00	0.615	0.723	4.11	(1)
Sun/Earth	1.00 ± 0.00	1.000	1.000	4.83	(1)
Sun/Mars	1.00 ± 0.00	1.881	1.524	5.96	(1)
Sun/Hung.	1.00 ± 0.00	2.72	1.95	6.75	(1)
Sun/Ceres	1.00 ± 0.00	4.61	2.77	8.04	(1)
Sun/Cyb.	1.00 ± 0.00	6.35	3.43	8.95	(1)
51 Peg	0.98 ± 0.12	0.01158	0.052	1.10	(2)
47 UMa	1.03 ± 0.09	3.020	2.049	6.91	(10)
70 Vir	1.12 ± 0.03	0.3195	0.433	3.18	(12)
HD 114762	0.82 ± 0.10	0.230	0.428	3.16	(13)
55 Cnc	0.90 ± 0.08	0.0404	0.126	1.72	(4)
Tau Boo	1.20 ± 0.12	0.00907	0.039	0.95	(4)
Ups And b	1.10 ± 0.12	0.0126	0.051	1.09	(4)
Ups And c	1.10 ± 0.12	0.663	0.714	4.08	(15)
Ups And d	1.10 ± 0.12	3.474	2.153	7.09	(15)
16 Cyg	1.05 ± 0.10	2.201	1.638	6.18	(8)
Rho CrB	1.00 ± 0.10	0.1085	0.227	2.30	(5)
HR 7875?	1.20 ± 0.12	0.116	0.211	2.22	(21)
Iota Hor	1.03 ± 0.02	0.876	0.897	4.58	(21)
GJ 876	0.32 ± 0.03	0.167	0.648	3.89	(9)
HD 187123	1.00 ± 0.10	0.00848	0.042	0.98	(3)
HD 210277	0.92 ± 0.11	1.196	1.191	5.27	(7)
HD 217107	0.96 ± 0.06	0.0195	0.074	1.32	(6)
HD 195019	0.98 ± 0.06	0.0501	0.138	1.79	(6)
HD 168443	0.84 ± 0.09	0.159	0.330	2.78	(7)
GJ 86	0.79 ± 0.08	0.0433	0.144	1.84	(11)
HD 75289	1.05 ± 0.10	0.00961	0.044	1.01	(14)
HD 130322	0.79 ± 0.05	0.0294	0.111	1.61	(17)
14 Her	1.06 ± 0.11	4.65	2.680	7.91	(18)
HD 209458	1.10 ± 0.05	0.00965	0.043	1.00	(19)
HD 192263	0.75 ± 0.05	0.0654	0.197	2.14	(16)
HD 37124	0.91 ± 0.10	0.424	0.601	3.75	(20)
HD 177830	1.00 ± 0.15	1.070	1.046	4.94	(20)
HD 134987	1.05 ± 0.10	0.712	0.772	4.25	(20)
HD 222582	1.00 ± 0.10	1.577	1.355	5.63	(20)
HD 10697	1.00 ± 0.10	2.965	2.064	6.94	(20)
GJ 3021	0.90 ± 0.09	0.366	0.549	3.58	(22)
HD 89744	1.40 ± 0.10	0.701	0.631	3.84	(22)
HD 12661	1.07 ± 0.10	0.724	0.771	4.24	(22)
HD 16141	1.01 ± 0.10	0.208	0.349	2.85	(22)
HD 46375	1.00 ± 0.10	0.00828	0.041	0.98	(22)

Table 1. Inner Solar System planets and extra-solar planets. The table gives, for each new exoplanet, the parent star mass M (in unit of $M_\odot$) and its uncertainty, the orbital period P(in years), the a/M ratio (in AU/$M_\odot$), given by $(P/M)^{2/3}$ from Kepler's third law, and the effective quantum number $\tilde{n} = 4.83\,(P/M)^{1/3} = 144/v$. Here a is the semi-major axis and v the average velocity, in km/s (see text).

n	obs.	dispersion	Student	rate	number
1	8	0.13	1.92	0.11	0.90
2	7	0.14	1.79	0.12	0.85
3	4	0.16	1.56	0.20	0.80
4	6	0.18	1.39	0.20	1.20
5	4	0.21	1.19	0.30	1.20
6	2	0.24	1.04	0.40	0.80
7	3	0.26	0.96	0.40	1.20
8	1	0.29	0.86	0.55	0.55

Table 2. Error analysis of the semi-major axis distribution of exoplanets. The table gives: the quantum number n; the observed number of exoplanets in each orbital; the expected total dispersion on $\tilde{n}$ obtained from a combination of the estimated intrinsic dispersion $\sigma_i = 0.125$ and of the dispersion coming from the error on the star mass, $\sigma_m = 0.033\,n$; the value of the Student t variable which corresponds to the observed number of exoplanets, for a difference $\delta n = \pm 0.25$; the corresponding expected rate of discrepant exoplanets; the final number of expected discrepant exoplanets in each orbital.