Non-differentiable space-time and scale relativity*

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Abstract

The theory of scale relativity consists of developing the consequences of the withdrawal of the hypothesis of space-time differentiability. Space-time acquires a fractal geometry, namely, it becomes explicitly dependent on the observation scale. The space-time resolutions are redefined as characterizing the state of scale of the reference system, then we set the principle of scale relativity, according to which the laws of nature should be valid whatever this state. The structures that are described in the scale space as coming under this principle induce a mechanics of the quantum type in the standard space of positions and instants. Various levels of the theory of scale relativity can be taken into account (Galilean, special then general scale relativity, coupling between scales and motion, quantum scale theory), and allow one to suggest possible generalizations of presently existing theories. We shall focus the present contribution on the detailed demonstration of the Schrödinger equation, then we consider possible applications to the problem of formation and evolution of gravitational structures.

Résumé: La théorie de la relativité d'échelle consiste à développer les conséquences de l'abandon de l'hypothèse de différentiabilité de l'espace-temps. Celui-ci acquiert un caractère fractal, c'est-à-dire explicitement dépendant des résolutions. On redéfinit les résolutions spatio-temporelles comme caractérisant l'état d'échelle du référentiel, puis on postule un principe de relativité d'échelle, suivant lequel les lois de la nature doivent être valides quel que soit cet état. Les structures décrites dans l'espace des échelles comme satisfaisant à ce principe induisent une mécanique de type quantique dans l'espace des positions et des instants. Plusieurs niveaux de description de la théorie relativiste d'échelle (galiléenne, einsteinienne restreinte puis générale, couplage entre échelle et mouvement, enfin elle-même quantique) peuvent être pris en compte et permettent de proposer des généralisations des théories existantes. On se concentrera dans la présente contribution sur la démonstration détaillée de l'équation de Schrödinger, puis on évoquera des applications possibles au problème de la formation et de l'évolution des structures gravitationnelles.

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1 Introduction

The theory of scale relativity of scale consists of applying the principle of relativity to transformations of scale (in particular to the transformations of the spatiotemporal resolutions). In the formulation of Einstein [1], the principle of relativity consists in requiring that the laws of nature are valid in any system of coordinates, whatever is its state. Since Galileo, this principle had been applied to the states of position (origin and orientation of axes) and of motion of the system of coordinates (velocity, acceleration), states which have the property to be never definable in a absolute way, but only in a relative way. The state of a reference system can be defined only with regard to another system.

It is the same as regards the changes of scale. The scale of a system can be defined only with regard to another system, and so owns the fundamental property of relativity: only scale ratios do have a physical meaning, never an absolute scale. In the new approach, one reinterprets the resolutions, not only as a property of the measuring device and / or of the measured system, but more generally as an intrinsic property of space-time, characterizing the state of scale of the reference system in the same way as velocity characterizes its state of movement. The principle of relativity of scale then consists in requiring that the fundamental laws of nature apply whatever the state of scale of the coordinate system.

What is the motivation to add such a first principle to fundamental physics? It becomes imperative from the very moment one wants to generalize the current description of space and time. The present description is usually reduced to differentiable manifolds (even though singularities are possible at certain particular points). So a way of generalization of current physics consists in trying to abandon the hypothesis of differentiability of spatiotemporal coordinates. The main consequence of such a giving up is that space-time becomes fractal, namely, in Mandelbrot's definition [2, 3], it acquires an explicit scale dependence (more precisely, it becomes scale-divergent) in terms of the spatiotemporal resolutions [4, 5].

The introduction of non-differentiable trajectories in quantum mechanics dates back to pioneering works by Feynman [6] and was also underlying the various attempts of construction of a stochastic mechanics [7, 8] (which have however now been shown to be in contradiction with quantum mechanics [9]). The proposal that is developed here is different, since it is not based on a fractal description of trajectories, but of space-time itself [10, 11, 12, 13, 14, 15, 16]. Our aim is therefore to recover the trajectories as geodesics of the non-differentiable space-time [4].

The theory of scale relativity is constructed by completing the standard laws of classical physics (motion in space / displacement in space-time) by new scale laws (in which the space-time resolutions are used as intrinsic variables). We hope such a stage of the theory to be only provisional, and the motion and scale laws to be treated on the same footing in the final theory. However, before reaching such a goal, one must realize that the various possible combinations of scale laws and motion laws lead to a large number of sub-sets of the theory to be developed. Indeed, three domains of the theory are first to be considered:

(i) *pure scale-laws*: description of the internal structures of a non-differential space-time at a given point / event;

(ii) *induced effects of scale laws on the equations of motion*: generation of the quantum mechanics as mechanics on a nondifferentiable space-time;

(iii) *scale-motion coupling*: effects of dilations induced by displacements, that we tentatively interpret as gauge fields (only the case of the electromagnetic field has been considered up to now) [17, 18].

Several levels of the description of scale laws (point i) can be considered. These levels are quite parallel to that of the historical development of the theory of motion:

(i1) *Galilean scale-relativity*: standard laws of dilation, that have the structure of a Galileo group (fractal power law with constant fractal dimension). When the fractal dimension of trajectories is $D = 2$, the induced motion laws are that of standard quantum mechanics [4, 18].

(i2) *Special scale-relativity*: generalization of the laws of dilation to a Lorentzian form [15]. The fractal dimension becomes a variable, and plays the role of a fifth dimension. An impassable length-time scale, invariant under dilations, appears in the theory; it replaces the zero, owns all its physical properties (an infinite energy-momentum would be needed to reach it), and plays for scale laws the same role as played by the velocity of light for motion.

(i3) *Scale-dynamics*: while the first two cases correspond to "scale freedom", one can also consider distorsion from strict self-similary that would come from the effect of a "scale-force" or "scale-field" [19, 20].

(i4) *General scale-relativity*: in analogy with the field of gravitation being ultimately attributed to the geometry of space-time, a future more profound description of the scale-field could be done in terms of Riemann geometry of the fifth-dimensional scale space.

(i5) *Quantum scale-relativity*: the above cases assume differentiability of the scale transformations. If one assumes them to be continuous but, as we have assumed for space-time, non-differentiable, one is confronted for scale laws to the same conditions that lead to quantum mechanics in space-time. One may therefore conjecture that quantum mechanical scale laws could be constructed in a future work.

The possible complication of the theory becomes apparent when one realizes that these various levels of the description of scale laws will lead to different levels of induced dynamics (point ii) and scale-motion coupling (iii), and that other sublevels are to be considered, depending on the status of motion laws (non-relativistic, special-relativistic, general-relativistic).

In the present contribution, we shall focus on point (i1), by giving a detailed and improved demonstration of the Schrödinger equation.

2 Schrödinger equation: detailed derivation from non-differentiability

2.1 Scale invariance and Galilean scale relativity

Consider a non-differentiable (fractal) curvilinear coordinate $\mathcal{L}(x, \varepsilon)$, that depends on some space-time variables x and on the resolution ε. Such a coordinate generalizes to non-differentiable and fractal space-times the concept of curvilinear coordinates introduced for curved Riemannian space-times in Einstein's general relativity [4]. $\mathcal{L}(x, \varepsilon)$, being differentiable when $\varepsilon \neq 0$, can be the solution of differential equations involving the derivatives of $\mathcal{L}$ with respect to both x and ε.

Let us apply an infinitesimal dilation $\varepsilon \to \varepsilon' = \varepsilon(1 + d\rho)$ to the resolution. Being, at this stage, interested in pure scale laws, we omit the x dependence in order to simplify the notation and we obtain, at first order,

$$\mathcal{L}(\varepsilon') = \mathcal{L}(\varepsilon + \varepsilon\, d\rho) = \mathcal{L}(\varepsilon) + \frac{\partial \mathcal{L}(\varepsilon)}{\partial \varepsilon}\, \varepsilon\, d\rho = (1 + \tilde{D}\; d\rho)\, \mathcal{L}(\varepsilon), \tag{1}$$

where $\tilde{D}$ is, by definition, the dilation operator. The identification of the two last members of this equation yields

$$\tilde{D} = \varepsilon \,\frac{\partial}{\partial \varepsilon} = \frac{\partial}{\partial \ln \varepsilon}\ . \tag{2}$$

This well-known form of the infinitesimal dilation operator shows that the "natural" variable for the resolution is $\ln \varepsilon$, and that the expected new differential equations will indeed involve quantities as $\partial \mathcal{L}(x, \varepsilon)/\partial \ln \varepsilon$. The renormalization group equations, in the multi-scale-of-length approach proposed by Wilson [21, 22], already describe such a scale dependence. The scale-relativity approach allows one to suggest more general forms for these scale groups.

The simplest renormalization group-like equation states that the variation of $\mathcal{L}$ under an infinitesimal scale transformation $d \ln \varepsilon$ depends only on $\mathcal{L}$ itself. We thus write

$$\frac{\partial \mathcal{L}(x, \varepsilon)}{\partial \ln \varepsilon} = \beta(\mathcal{L}). \tag{3}$$

Still looking for the simplest form of such an equation, we expand $\beta(\mathcal{L})$ in powers of $\mathcal{L}$. We obtain, to the first order, the linear equation

$$\frac{\partial \mathcal{L}(x, \varepsilon)}{\partial \ln \varepsilon} = a + b\mathcal{L}\ , \tag{4}$$

of which the solution is

$$\mathcal{L}(x, \varepsilon) = \mathcal{L}_0(x)\ \left[1 + \zeta(x) (\frac{\lambda}{\varepsilon})^{-b}\right], \tag{5}$$

where $\lambda^{-b}\zeta(x)$ is an integration constant and $\mathcal{L}_0 = -a/b$.

The scale dimension, $\delta = D - D_T$, where D is the fractal dimension and D_T the topological dimension, is defined, following Mandelbrot [2, 3], as

$$\delta = \frac{d \ln \mathcal{L}}{d \ln(\lambda/\varepsilon)} \ . \tag{6}$$

When δ is constant, one obtains an asymptotic power law resolution dependence

$$\mathcal{L}(x, \varepsilon) = \mathcal{L}_0(x) \left(\frac{\lambda}{\varepsilon}\right)^{\delta} . \tag{7}$$

Let us now check that such a simple self-similar scaling law does come under the principle of relativity extended to scale transformations of the resolutions. The involved quantities transform, under a scale transformation $\varepsilon \to \varepsilon'$, as

$$\ln \frac{\mathcal{L}(\varepsilon')}{\mathcal{L}_0} = \ln \frac{\mathcal{L}(\varepsilon)}{\mathcal{L}_0} + \delta(\varepsilon) \ln \frac{\varepsilon}{\varepsilon'} \ , \tag{8}$$

$$\delta(\varepsilon') = \delta(\varepsilon). \tag{9}$$

These transformations have exactly the mathematical structure of the Galileo group (applied here to scale rather than motion), as confirmed by the dilation composition law, $\varepsilon \to \varepsilon' \to \varepsilon''$, which writes

$$\ln \frac{\varepsilon''}{\varepsilon} = \ln \frac{\varepsilon'}{\varepsilon} + \ln \frac{\varepsilon''}{\varepsilon'} \ , \tag{10}$$

and is therefore similar to the law of composition of velocities. It is worth noting that Eq. (5) gives, in addition, a transition from a fractal to a non-fractal behavior at scales larger than some transition scale λ. In other words, contrarily to the case of motion laws, for which the invariance group is universal, the scale group symmetry is broken beyond some (relative) transition scale.

2.2 Lagrangian approach to scale laws

The Lagrangian approach can be used in the scale space in order to obtain physically relevant generalizations of the above simplest (scale-invariant) laws. In this aim, we are led to reverse the definition and meaning of the variables. Namely, the scale dimension δ becomes a primary variable that plays, for scale laws, the same role as played by time in motion laws.

The resolution, ε, can therefore be defined as a derived quantity in terms of the fractal coordinate $\mathcal{L}$ and of the scale dimension δ

$$\tilde{V} = \ln(\frac{\lambda}{\varepsilon}) = \frac{d \ln \mathcal{L}}{d\delta} \ . \tag{11}$$

A scale Lagrange function $\tilde{L}(\ln \mathcal{L}, \tilde{V}, \delta)$ is introduced, from which a scale action is constructed

$$\tilde{S} = \int_{\delta_1}^{\delta_2} \tilde{L}(\ln \mathcal{L}, \tilde{V}, \delta) \ d\delta. \tag{12}$$

The application of the action principle yields a scale Euler-Lagrange equation that writes

$$\frac{d}{d\delta}\frac{\partial \tilde{L}}{\partial \tilde{V}} = \frac{\partial \tilde{L}}{\partial \ln \mathcal{L}} \,. \tag{13}$$

In analogy with the physics of motion, the simplest possible form for the Lagrange function is a quadratic dependence on the "scale velocity", $\tilde{V}$, (i.e., $\tilde{L} \propto \tilde{V}^2$) and the absence of any "scale force" (i.e., $\partial\tilde{L}/\partial \ln \mathcal{L} = 0$), which is the equivalent for scale of what inertia is for motion. The Euler-Lagrange equation becomes in this case

$$\frac{d\tilde{V}}{d\delta} = 0 \Rightarrow \tilde{V} = \text{const.} \tag{14}$$

The constancy of $\tilde{V} = \ln(\lambda/\varepsilon)$ means that it is independent of the "scale time" δ. Equation (11) can therefore be integrated to give the usual power law behavior, $\mathcal{L} = \mathcal{L}_0(\lambda/\varepsilon)^\delta$. This reversed viewpoint has several advantages which allow a full implementation of the principle of scale relativity:

(i) The scale dimension δ is given its actual status of "scale time" and the logarithm of the resolution, $\tilde{V}$, its status of "scale velocity" (see Eq. (11)). This is in accordance with its scale-relativistic definition, in which it characterizes the "state of scale" of the reference system, in the same way as the velocity $v = dx/dt$ characterizes its state of motion.

(ii) This leaves open the possibility of generalizing our formalism to the case of four independent space-time resolutions, $\tilde{V}^\mu = \ln(\lambda^\mu/\varepsilon^\mu) = d\ln\mathcal{L}^\mu/d\delta$. (Let us however remark from now, to be more specific, that the genuine nature of resolutions is tensorial, $\varepsilon_\mu^\nu = \varepsilon_\mu\varepsilon^\nu = \rho_{\mu\lambda}\varepsilon^\nu\varepsilon^\lambda$ and involves correlation coefficients, as in a variance-covariance matrix).

(iii) Scale laws more general than the simplest self-similar ones can be derived from more general scale Lagrangians [19].

2.3 Transition from non-differentiability (small scales) to differentiability (large scales)

Strictly, the non-differentiability of the coordinates means that the velocity

$$V = \frac{dX}{dt} = \lim_{dt\to 0} \frac{X(t+dt) - X(t)}{dt} \tag{15}$$

is undefined. Namely, when dt tends to zero, either the ratio dX/dt tends to infinity, or it fluctuates without reaching any limit. However, as recalled in the introduction, continuity and non-differentiability imply an explicit scale dependence of the various physical quantities, and therefore of the velocity, V. We therefore apply to the velocity and to the differential element, now interpreted as a resolution, the reasoning applied to the fractal function $\mathcal{L}$ in Sec. 2.1. We obtain the solution

$$V = v + w = v\,\left[1 + \zeta\left(\frac{\tau}{dt}\right)^{1-1/D}\right]. \tag{16}$$

This means that the velocity is now the sum of two independent terms of different orders, since their ratio v/w is, from the standard viewpoint, infinitesimal. In analogy with the real and imaginary parts of a complex number, we suggest [23] to call v the "classical part" of the velocity, $v = \mathcal{C}\ell\langle V\rangle$, (see below the definition of the classical part operator $C\ell\langle\ \ \rangle$). The new component w is an explicitely scale-dependent fractal fluctuation (which would be infinite from the standard point of view where one makes $dt \to \infty$ and τ and ζ are chosen such that $\mathcal{C}\ell\langle\zeta\rangle = 0$ and $\mathcal{C}\ell\langle\zeta^2\rangle = 1$.

We recognize here the combination of a typical fractal behavior, with a fractal dimension D, and of a breaking of the scale symmetry at the scale transition τ. As we shall see, in what follows, τ will be identified with the de Broglie scale of the system ($\tau = \hbar/E$), since $V \approx v$, when $dt \gg \tau$ (classical behavior), and $V \approx w$, when $dt \ll \tau$ (fractal behavior). Recalling that $D = 2$ plays the role of a critical dimension [4], we stress that, in the asymptotic scaling domain, $w \propto (dt/\tau)^{-1/2}$, in agreement with Ref. [6] for quantum paths, which allows to identify the fractal domain with the quantum one.

The above description strictly applies to an individual fractal trajectory. Now, one of the geometric consequences of the non-differentiability and of the subsequent fractal character of space/space-time itself (not only of the trajectories) is that there is an infinity of fractal geodesics relating any couple of points of this fractal space. It has therefore been suggested [12] that the description of a quantum mechanical particle, including its property of wave-particle duality, could be reduced to the geometric properties of the set of fractal geodesics that corresponds to a given state of this "particle". In such an interpretation, we do not have to endow the "particle" with internal properties such as mass, spin or charge, since the "particle" is not identified with a point mass which would follow the geodesics, but its internal properties can simply be defined as geometric properties of the fractal geodesics themselves. As a consequence, any measurement is interpreted as a sorting out (or selection) of the geodesics of which the properties correspond to the resolution scale of the measuring device (as an example, if the "particle" has been observed at a given position with a given resolution, this means that the geodesics which pass through this domain have been selected) [4, 12].

The transition scale appearing in Eq. (16) yields two distinct behaviors of the system (particle) depending on the resolution at which it is considered. Equation (16) multiplied by dt gives the elementary displacement, dX, of the system as a sum of two infinitesimal terms of different orders

$$dX = dx + d\xi. \tag{17}$$

Here $d\xi$ represents the fractal fluctuations and $dx = \mathcal{C}\ell\langle dX\rangle$ is the "classical" or "large-scale" value. They are defined as

$$dx = v\ dt, \tag{18}$$

$$d\xi = \eta\sqrt{2\mathcal{D}}(dt^2)^{1/2D}, \tag{19}$$

which becomes, for $D=2$,

$$d\xi = \eta\sqrt{2\mathcal{D}}dt^{1/2}, \tag{20}$$

with $2\mathcal{D} = \tau_0 = \tau v^2$, $\mathcal{C}\ell\langle\eta\rangle = 0$ and $\mathcal{C}\ell\langle\eta^2\rangle = 1$. Owing to Eq. (16), we identify τ with the Einstein transition scale, $\hbar/E = \hbar/\frac{1}{2}mv^2$. Therefore, as we shall see further on, $2\mathcal{D} = \tau_0$ is a scalar quantity which can be identified with the Compton scale, $\hbar/mc$, i.e., it gives the mass of the particle up to fundamental constants.

Now, the Schrödinger, Klein-Gordon and Dirac equations give results applying to measurements performed on quantum objects, but achieved with classical devices, in the differentiable "large-scale" domain. The microphysical scale at which the physical systems under study are considered induces the selection of a bundle of geodesics, corresponding to the scale of the systems (see above), while the measurement process implies a smoothing out of the geodesic bundle coupled to a transition from the non-differentiable "small-scale" to the differentiable "large-scale" domain. We are therefore led to define an operator $\mathcal{C}\ell\langle\quad\rangle$, which we apply to the fractal variables or functions each time we are drawn to the classical domain where the dt behavior dominates.

Please note the improvement of the new definition in terms of the large-scale part [23] with respect to the previous interpretation in terms of an averaging process [4].

2.4 Differential-time symmetry breaking

Another consequence of the non-differentiable nature of space (space-time) is the breaking of local differential (proper) time reflection invariance. The derivative with respect to the time t of a differentiable function f can be written twofold

$$\frac{df}{dt} = \lim_{dt\to 0}\frac{f(t+dt)-f(t)}{dt} = \lim_{dt\to 0}\frac{f(t)-f(t-dt)}{dt}\,. \tag{21}$$

The two definitions are equivalent in the differentiable case. One passes from one to the other by the transformation $dt \leftrightarrow -dt$ (differential time reflection invariance), which is therefore an implicit discrete symmetry of differentiable physics. In the non-differentiable situation, both definitions fail, since the limits are no longer defined. In the new framework of scale relativity, the physics is related to the behavior of the function during the "zoom" operation on the time resolution δt, identified with the differential element dt. Two functions f'_+ and f'_- are therefore defined as explicit functions of t and dt

$$f'_+(t,dt) = \frac{f(t+dt,dt)-f(t,dt)}{dt}\,, \tag{22}$$

$$f'_-(t,dt) = \frac{f(t,dt)-f(t-dt,dt)}{dt}\,. \tag{23}$$

When applied to the space coordinates, these definitions yield, in the non-differentiable domain, two velocities that are fractal functions of the resolution,

$V_+[x(t), t, dt]$ and $V_-[x(t), t, dt]$. In order to go back to the classical domain and derive the classical velocities appearing in Eq. (18), we smooth out each fractal geodesic in the bundles selected by the zooming process with balls of radius larger than τ. This amounts to carry out a transition from the non-differentiable to the differentiable domain and leads to define two classical velocity fields now resolution-independent: $V_+[x(t), t, dt > \tau] = \mathcal{C}\ell\langle V_+[x(t), t, dt]\rangle = v_+[x(t), t]$ and $V_-[x(t), t, dt > \tau] = \mathcal{C}\ell\langle V_-[x(t), t, dt]\rangle = v_-[x(t), t]$. The important new fact appearing here is that, after the transition, there is no longer any reason for these two velocities to be equal. While, in standard mechanics, the concept of velocity was one-valued, we must introduce, for the case of a non-differentiable space, two velocities instead of one, even when going back to the classical domain.

A simple and natural way to account for this doubling consists in using complex numbers and the complex product. As we recall hereafter, this is the origin of the complex nature of the wave function of quantum mechanics, since this wave function can be identified with the exponential of the complex action that is naturally introduced in this framework. We shall now demonstrate that the choice of complex numbers to represent the two-valuedness of the velocity is a simplifying and "covariant" choice.

2.5 Covariant derivative operator

We are now lead to describe the elementary displacements for both processes, $dX_\pm$, as the sum of a $\mathcal{C}\ell$ part, $dx_\pm = v_\pm\ dt$, and a fluctuation about this $\mathcal{C}\ell$ part, $d\xi_\pm$, which is, by definition, of zero classical part, $\mathcal{C}\ell\langle d\xi_\pm\rangle = 0$

$$dX_+(t) = v_+\ dt + d\xi_+(t),$$
$$dX_-(t) = v_-\ dt + d\xi_-(t). \tag{24}$$

Considering first the large-scale displacements, large-scale forward and backward derivatives, d/dt_+ and d/dt_-, are defined, using the $\mathcal{C}\ell$ part extraction procedure. Applied to the position vector, x, they yield the forward and backward large-scale velocities

$$\frac{d}{dt_+}x(t) = v_+\ , \qquad \frac{d}{dt_-}x(t) = v_-\ . \tag{25}$$

As regards the fluctuations, the generalization to three dimensions of the fractal behavior of Eq. (19) writes (for $D = 2$)

$$\mathcal{C}\ell\langle d\xi_{\pm i}\ d\xi_{\pm j}\rangle = \pm 2\,\mathcal{D}\,\delta_{ij}\ dt \qquad i, j = x, y, z, \tag{26}$$

as the $d\xi(t)$'s are of null $\mathcal{C}\ell$ part and mutually independent. The Krönecker symbol, δ_{ij}, in Eq. (26), implies indeed that the $\mathcal{C}\ell$ part of every crossed product $\mathcal{C}\ell\langle d\xi_{\pm i}\ d\xi_{\pm j}\rangle$, with $i \neq j$, is null.

2.5.1 Origin of complex numbers in quantum mechanics

We now know that each component of the velocity takes two values instead of one. This means that each component of the velocity becomes a vector in a two-dimensional space, or, in other words, that the velocity becomes a two-index tensor. The generalization of the sum of these quantities is straighforward, but one also needs to define a generalized product.

The problem can be put in a general way: it amounts to find a generalization of the standard product that keeps its fundamental physical properties.

From the mathematical point of view, we are here exactly confronted to the well-known problem of the doubling of algebra (see, e.g., Ref. [24]). Indeed, the effect of the symmetry breaking $dt \leftrightarrow -dt$ (or $ds \leftrightarrow -ds$) is to replace the algebra $\mathcal{A}$ in which the classical physical quantities are defined, by a direct sum of two exemplaries of $\mathcal{A}$, i.e., the space of the pairs (a, b) where a and b belong to $\mathcal{A}$. The new vectorial space $\mathcal{A}^2$ must be supplied with a product in order to become itself an algebra (of doubled dimension). The same problem is asked again when one takes also into account the symmetry breakings $dx^\mu \leftrightarrow -dx^\mu$ and $x^\mu \leftrightarrow -x^\mu$ (see [23]): this leads to new algebra doublings. The mathematical solution to this problem is well-known: the standard algebra doubling amounts to supply $\mathcal{A}^2$ with the complex product. Then the doubling $\mathbb{R}^2$ of $\mathbb{R}$ is the algebra $\mathbb{C}$ of complex numbers, the doubling $\mathbb{C}^2$ of $\mathbb{C}$ is the algebra $\mathbb{H}$ of quaternions, the doubling $\mathbb{H}^2$ of quaternions is the algebra of Graves-Cayley octonions. The problem with algebra doubling is that the iterative doubling leads to a progressive deterioration of the algebraic properties. Namely, the quaternion algebra is non-commutative, while the octonion algebra is also non-associative. But an important positive result for physical applications is that the doubling of a metric algebra is a metric algebra [24].

These mathematical theorems fully justify the use of complex numbers, then of quaternions, in order to describe the successive doublings due to discrete symmetry breakings at the infinitesimal level, which are themselves more and more profound consequences of space-time non-differentiability.

However, we give in what follows complementary arguments of a physical nature, which show that the use of the complex product in the first algebra doubling have a simplifying and covariant effect (we use here the word "covariant" in the original meaning given to it by Einstein [1], namely, the requirement of the form invariance of fundamental equations).

In order to simplify the argument, let us consider the generalization of scalar quantities, for which the product law is the standard product in $\mathbb{R}$.

The first constraint is that the new product must remain an internal composition law. We also make the simplifying assumption that it remains linear in terms of each of the components of the two quantities to be multiplied.

The second physical constraint is that we recover the classical variables and the classical product at the classical limit. The mathematical equivalent of this constraint is the requirement that $\mathcal{A}$ still be a sub-algebra of $\mathcal{A}^2$. Therefore we identify $a_0 \in \mathcal{A}$ with $(a_0, 0)$ and we set $(0, 1) = \alpha$. This allows us to write the new two-dimensional vectors in the simplified form $a = a_0 + a_1 \alpha$, so that the product now writes

$$c = (a_0 + a_1 \alpha)\,(b_0 + b_1 \alpha) = a_0 b_0 + a_1 b_1 \alpha^2 + (a_0 b_1 + a_1 b_0)\alpha. \tag{27}$$

The problem is now reduced to find α^2, which is defined by only two coefficients

$$\alpha^2 = \omega_0 + \omega_1 \alpha. \tag{28}$$

Let us now come back to the beginning of our construction. We have introduced two elementary displacements, a forward and a backward one, each of them made of two terms, a $\mathcal{C}\ell$ part and a fluctuation (see Eq. (24))

$$\begin{aligned} dX_+(t) &= v_+ \; dt + d\xi_+(t), \\ dX_-(t) &= v_- \; dt + d\xi_-(t). \end{aligned} \tag{29}$$

Therefore, one can define velocity fluctuations $w_+ = d\xi_+/dt$ and $w_- = d\xi_-/dt$, then a complete velocity in the doubled algebra [25]

$$\mathcal{V} + \mathcal{W} = \left(\frac{v_+ + v_-}{2} - \alpha \, \frac{v_+ - v_-}{2} \right) + \left(\frac{w_+ + w_-}{2} - \alpha \, \frac{w_+ - w_-}{2} \right) . \tag{30}$$

We shall see in what follows that a Lagrange function can be introduced in terms of the new two-valued tool, that leads to a conserved form for the Euler-Lagrange equations. In the end, the Schrödinger equation is obtained as their integral. Now, from the covariance principle, the Lagrange function in the Newtonian case should strictly be written:

$$\mathcal{L} = \frac{1}{2} m \, \mathcal{C}\ell \langle (\mathcal{V} + \mathcal{W})^2 \rangle = \frac{1}{2} m \; \left(\mathcal{C}\ell \langle \mathcal{V}^2 \rangle + \mathcal{C}\ell \langle \mathcal{W}^2 \rangle \right) \tag{31}$$

We have $\mathcal{C}\ell\langle \mathcal{W} \rangle = 0$, by definition, and $\mathcal{C}\ell\langle \mathcal{V}\mathcal{W} \rangle = 0$, because they are mutually independent. But what about $\mathcal{C}\ell\langle \mathcal{W}^2 \rangle$? The presence of this term would greatly complicate all the subsequent developments toward the Schrödinger equation, since it would imply a fundamental divergence of non-relativistic quantum mechanics. Let us expand it:

$$\begin{aligned} 4 \, \mathcal{C}\ell\langle \mathcal{W}^2 \rangle &= \mathcal{C}\ell\langle [(w_+ + w_-) - \alpha \, (w_+ - w_-)]^2 \rangle \\ &= \mathcal{C}\ell\langle (w_+^2 + w_-^2)(1 + \alpha^2) - 2\alpha(w_+^2 - w_-^2) + 2 w_+ w_- (1 - \alpha^2) \rangle \end{aligned} \tag{32}$$

Since $\mathcal{C}\ell\langle w_+^2 \rangle = \mathcal{C}\ell\langle w_-^2 \rangle$ and $\mathcal{C}\ell\langle w_+ w_- \rangle = 0$ (they are mutually independent), we finally find that $\mathcal{C}\ell\langle \mathcal{W}^2 \rangle$ can only vanish provided

$$\alpha^2 = -1, \tag{33}$$

namely, $\alpha = i$, the imaginary. Therefore we have shown that the choice of the complex product in the algebra doubling plays an essential physical role, since it allows to suppress what would be additional infinite terms in the final equations of motion.

2.5.2 Complex velocity

We now combine the forward and backward derivatives to obtain a complex derivative operator, that allows us to recover local differential time reversibility in terms of the new complex process [4]:

$$\frac{\acute{d}}{dt} = \frac{1}{2} \left(\frac{d}{dt_+} + \frac{d}{dt_-} \right) - \frac{i}{2} \left(\frac{d}{dt_+} - \frac{d}{dt_-} \right) . \tag{34}$$

Applying this operator to the position vector yields a complex velocity

$$\mathcal{V} = \frac{\acute{d}}{dt} x(t) = V - iU = \frac{v_+ + v_-}{2} - i\,\frac{v_+ - v_-}{2}\ . \tag{35}$$

The minus sign in front of the imaginary term is chosen here in order to obtain the Schrödinger equation in terms of ψ. The reverse choice would give the Schrödinger equation for the complex conjugate of the wave function $\psi^\dagger$, and would be therefore physically equivalent.

The real part, V, of the complex velocity, $\mathcal{V}$, represents the standard classical velocity, while its imaginary part, $-U$, is a new quantity arising from non-differentiability. At the usual classical limit, $v_+ = v_- = v$, so that $V = v$ and $U = 0$.

2.5.3 Complex time derivative operator

Contrary to what happens in the differentiable case, the total derivative with respect to time of a fractal function $f(x(t), t)$ of integer fractal dimension contains finite terms up to higher order [26]

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\frac{dX_i}{dt} + \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}\frac{dX_i dX_j}{dt} + \frac{1}{6}\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}\frac{dX_i dX_j dX_k}{dt} + \ldots \tag{36}$$

Note that it has been shown by Kolwankar and Gangal [27] that, if the fractal dimension is not an integer, a fractional Taylor expansion can also be defined, using the local fractional derivative (however, see [28] about the physical relevance of this tool).

In our case, a finite contribution only proceeds from terms of D-order, while lesser-order terms yield an infinite contribution and higher-order ones are negligible. Therefore, in the special case of a fractal dimension $D = 2$, the total derivative writes

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \nabla f.\frac{dX}{dt} + \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}\frac{dX_i dX_j}{dt}\ . \tag{37}$$

Usually the term $dX_i dX_j/dt$ is infinitesimal, but here its $\mathcal{C}\ell$ part reduces to $\mathcal{C}\ell\langle d\xi_i\ d\xi_j\rangle/dt$. Therefore, thanks to Eq. (26), the last term of the large-scale part of Eq. (37) amounts to a Laplacian, and we can write

$$\frac{df}{dt_\pm} = \left(\frac{\partial}{\partial t} + v_\pm.\nabla \pm \mathcal{D}\Delta\right) f\ . \tag{38}$$

Substituting Eqs. (38) into Eq. (34), we finally obtain the expression for the complex time derivative operator [4]

$$\frac{\acute{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\mathcal{D}\Delta\ . \tag{39}$$

The passage from standard classical (almost everywhere differentiable) mechanics to the new non-differentiable theory can now be implemented by replacing the standard time derivative d/dt by the new complex operator $\acute{d}/dt$ [4]. In other words, this means that $\acute{d}/dt$ plays the role of a "covariant derivative operator" (in analogy with the covariant derivative $D_j A^k = \partial_j A^k + \Gamma^k_{jl} A^l$ replacing $\partial_j A^k$ in Einstein's general relativity).

2.6 Covariant mechanics induced by scale laws

Let us now summarize the main steps by which one may generalize the standard classical mechanics using this covariance. We assume that the large-scale part of any mechanical system can be characterized by a Lagrange function $\mathcal{L}(x, \mathcal{V}, t)$, from which an action $\mathcal{S}$ is defined

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}(x, \mathcal{V}, t) dt. \tag{40}$$

In this expression, we have combined the forward and backward velocities in terms of a unique complex velocity. We have already given arguments, in the previous section, according to which this choice is a simplifying and covariant choice. We shall now support this conclusion by demonstrating hereafter that it indeed allows us to conserve the standard form of the Euler-Lagrange equations.

In a general way, the Lagrange function is expected to be a function of the variables x and their time derivatives $\dot{x}$. We have found that the number of velocity components $\dot{x}$ is doubled, so that we are led to write

$$L = L(x, \dot{x}_+, \dot{x}_-, t). \tag{41}$$

Instead, we have made the choice to write the Lagrange function as $L = L(x, \mathcal{V}, t)$. We now justify this choice by the covariance principle. Re-expressed in terms of $\dot{x}_+$ and $\dot{x}_-$, the Lagrange function writes

$$L = L\left(x, \frac{1-i}{2}\,\dot{x}_+ + \frac{1+i}{2}\,\dot{x}_-, t\right). \tag{42}$$

Therefore we obtain

$$\frac{\partial L}{\partial \dot{x}_+} = \frac{1-i}{2}\,\frac{\partial L}{\partial \mathcal{V}} \quad ; \quad \frac{\partial L}{\partial \dot{x}_-} = \frac{1+i}{2}\,\frac{\partial L}{\partial \mathcal{V}}, \tag{43}$$

while the new covariant time derivative operator writes

$$\frac{\acute{d}}{dt} = \frac{1-i}{2}\,\frac{d}{dt_+} + \frac{1+i}{2}\,\frac{d}{dt_-}\,. \tag{44}$$

Let us write the stationary action principle in terms of the Lagrange function of Eq. (41)

$$\delta S = \delta \int_{t_1}^{t_2} L(x, \dot{x}_+, \dot{x}_-, t)\; dt = 0. \tag{45}$$

It becomes

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x}\, \delta x + \frac{\partial L}{\partial \dot{x}_+}\, \delta \dot{x}_+ + \frac{\partial L}{\partial \dot{x}_-}\, \delta \dot{x}_- \right) dt = 0. \tag{46}$$

Since $\delta \dot{x}_+ = d(\delta x)/dt_+$ and $\delta \dot{x}_- = d(\delta x)/dt_-$, it takes the form

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x}\, \delta x + \frac{\partial L}{\partial \mathcal{V}} \left[\frac{1-i}{2} \frac{d}{dt_+} + \frac{1+i}{2} \frac{d}{dt_-} \right] \delta x \right) dt = 0, \tag{47}$$

i.e.,

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x}\, \delta x + \frac{\partial L}{\partial \mathcal{V}} \frac{\acute{d}}{dt}\, \delta x \right) dt = 0. \tag{48}$$

The subsequent demonstration of the Lagrange equations relies on an integration by part. Now such an operation involves the Leibniz rule for the covariant derivative operator $\acute{d}/dt$. Since $\acute{d}/dt = \partial/dt + \mathcal{V}\,.\,\nabla - i\mathcal{D}\Delta$ is a linear combination of first and second order derivatives, the same is true of its Leibniz rule. This implies the appearance of an additional term in the expression for the derivative of a product [31]:

$$\frac{\acute{d}}{dt} \left[\frac{\partial L}{\partial \mathcal{V}}\,.\,\delta x \right] = \frac{\acute{d}}{dt} \frac{\partial L}{\partial \mathcal{V}}\,.\,\delta x + \frac{\partial L}{\partial \mathcal{V}}\,.\,\frac{\acute{d}}{dt} \delta x - 2i\,\mathcal{D}\,\nabla \frac{\partial L}{\partial \mathcal{V}}\,.\,\nabla \delta x. \tag{49}$$

Since $\delta x(t)$ is not a function of x, the additional term vanishes. After having defined a new integration as the inverse of the covariant derivation, i.e. $\int \acute{d} f = f$, the integral reduces to:

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} - \frac{\acute{d}}{dt} \frac{\partial L}{\partial \mathcal{V}} \right) \delta x\, dt = 0. \tag{50}$$

Finally the Euler-Lagrange equations write

$$\frac{\acute{d}}{dt} \frac{\partial L}{\partial \mathcal{V}} = \frac{\partial L}{\partial x}. \tag{51}$$

Therefore, thanks to the transformation $d/dt \to \acute{d}/dt$, they take exactly their standard classical form. This result reinforces the identification of our tool with a "quantum-covariant" representation, since, as we have shown in previous works and as we recall in what follows, this Euler-Lagrange equation can be integrated in the form of a Schrödinger equation.

Assuming homogeneity of space in the mean leads to define a generalized complex momentum given by

$$\mathcal{P} = \frac{\partial \mathcal{L}}{\partial \mathcal{V}}. \tag{52}$$

If we now consider the action as a functional of the upper limit of integration in Eq. (40), the variation of the action from a trajectory to another nearby trajectory yields a generalization of another well-known relation of standard mechanics:

$$\mathcal{P} = \nabla \mathcal{S}. \tag{53}$$

As concerns the generalized energy, its expression involves an additional term [31, 43]: namely it write for a Newtonian Lagrange function and in the absence of exterior potential, $\mathcal{E} = (1/2)m(\mathcal{V}^2 - 2i\mathcal{D}\,\mathrm{div}\mathcal{V})$.

2.7 Generalized Newton-Schrödinger Equation

Let us now specialize our study, and consider Newtonian mechanics, i.e., the general case when the structuring external scalar field is described by a potential energy Φ. The Lagrange function of a closed system, $L = \frac{1}{2}mv^2 - \Phi$, is generalized, in the large-scale domain, as $\mathcal{L}(x, \mathcal{V}, t) = \frac{1}{2}m\mathcal{V}^2 - \Phi$. The Euler-Lagrange equations keep the form of Newton's fundamental equation of dynamics

$$m\frac{\acute{d}}{dt}\mathcal{V} = -\nabla\Phi, \tag{54}$$

which is now written in terms of complex variables and complex operators.

In the case when there is no external field, the covariance is explicit, since Eq. (54) takes the form of the equation of inertial motion

$$\acute{d}\mathcal{V}/dt = 0, \tag{55}$$

in analogy with what happens in general relativity, where the equivalence principle of gravitation and inertia leads to a strong covariance principle, expressed by the fact that one may always find a coordinate system in which the metric is locally Minkowskian. This means that, in this coordinate system, the covariant equation of motion of a free particle is that of inertial motion $Du_\mu = 0$ in terms of the general-relativistic covariant derivative D and four-vector u_μ. The expansion of the covariant derivative subsequently transforms this free-motion equation in a local geodesic equation in a gravitational field.

The covariance induced by scale effects leads to an analogous transformation of the equation of motions, which, as we show below, become the Schrödinger equation, (then the Klein-Gordon and Dirac equations in the motion-relativistic case), which we are therefore allowed to consider as local a geodesic equation.

In both cases, with or without external field, the complex momentum $\mathcal{P}$ reads

$$\mathcal{P} = m\mathcal{V}, \tag{56}$$

so that, from Eq. (53), the complex velocity $\mathcal{V}$ appears as a gradient, namely the gradient of the complex action

$$\mathcal{V} = \nabla\mathcal{S}/m. \tag{57}$$

We now introduce a complex wave function ψ which is nothing but another expression for the complex action $\mathcal{S}$

$$\psi = e^{i\mathcal{S}/\mathcal{S}_0}. \tag{58}$$

The factor $\mathcal{S}_0$ has the dimension of an action (i.e., an angular momentum) and must be introduced for dimensional reasons. We show in what follows, that, when this formalism is applied to microphysics, S_0 is nothing but the fundamental constant $\hbar$. The function ψ is related to the complex velocity appearing in Eq. (57) as follows

$$\mathcal{V} = -i\,\frac{\mathcal{S}_0}{m}\,\nabla(\ln\psi). \tag{59}$$

We have now at our disposal all the mathematical tools needed to write the fundamental equation of dynamics of Eq. (54) in terms of the new quantity ψ. It takes the form

$$i\mathcal{S}_0\,\frac{\acute{d}}{dt}(\nabla\ln\psi) = \nabla\Phi. \tag{60}$$

Now one should be aware that $\acute{d}$ and ∇ do not commute. However, as we shall see in the following, there is a particular choice of the arbitrary constant $\mathcal{S}_0$ for which $\acute{d}(\nabla\ln\psi)/dt$ is nevertheless a gradient.

Replacing $\acute{d}/dt$ by its expression, given by Eq. (39), yields

$$\nabla\Phi = i\mathcal{S}_0\left(\frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\mathcal{D}\Delta\right)(\nabla\ln\psi), \tag{61}$$

and replacing once again $\mathcal{V}$ by its expression in Eq. (59), we obtain

$$\nabla\Phi = i\mathcal{S}_0\left[\frac{\partial}{\partial t}\nabla\ln\psi - i\left\{\frac{\mathcal{S}_0}{m}(\nabla\ln\psi.\nabla)(\nabla\ln\psi) + \mathcal{D}\Delta(\nabla\ln\psi)\right\}\right]. \tag{62}$$

Consider now the remarkable identity

$$(\nabla\ln f)^2 + \Delta\ln f = \frac{\Delta f}{f}\;, \tag{63}$$

which proceeds from the following tensorial derivation

$$\begin{aligned}
\partial_\mu\partial^\mu\ln f + \partial_\mu\ln f\,\partial^\mu\ln f &= \partial_\mu\frac{\partial^\mu f}{f} + \frac{\partial_\mu f}{f}\frac{\partial^\mu f}{f} \\
&= \frac{f\partial_\mu\partial^\mu f - \partial_\mu f\partial^\mu f}{f^2} + \frac{\partial_\mu f\partial^\mu f}{f^2} \\
&= \frac{\partial_\mu\partial^\mu f}{f}\;.
\end{aligned} \tag{64}$$

When we apply this identity to ψ and take its gradient, we obtain

$$\nabla\left(\frac{\Delta\psi}{\psi}\right) = \nabla[(\nabla\ln\psi)^2 + \Delta\ln\psi]. \tag{65}$$

The second term in the right-hand side of this expression can be transformed, using the fact that ∇ and Δ commute, i.e.,

$$\nabla\Delta = \Delta\nabla. \tag{66}$$

The first term can also be transformed thanks to another remarkable identity

$$\nabla(\nabla f)^2 = 2(\nabla f.\nabla)(\nabla f), \tag{67}$$

that we apply to $f = \ln \psi$. We finally obtain

$$\nabla \left(\frac{\Delta \psi}{\psi} \right) = 2(\nabla \ln \psi.\nabla)(\nabla \ln \psi) + \Delta(\nabla \ln \psi). \tag{68}$$

We recognize, in the right-hand side of this equation, the two terms of Eq. (62), which were respectively in factor of $\mathcal{S}_0/m$ and $\mathcal{D}$. Therefore, the particular choice

$$\mathcal{S}_0 = 2m\mathcal{D} \tag{69}$$

allows us to simplify the right-hand side of Eq. (62). The simplification is twofold: (i) several complicated terms are compacted into a simple one; (ii) the final remaining term is a gradient, which means that the fundamental equation of dynamics can now be integrated in a universal way. The wave function in Eq. (58) is therefore defined as

$$\psi = e^{i\mathcal{S}/2m\mathcal{D}}, \tag{70}$$

and it is solution of the fundamental equation of dynamics, Eq. (54), which we write

$$\frac{d\!\!/}{dt}\mathcal{V} = -2\mathcal{D}\nabla \left\{ i\frac{\partial}{\partial t} \ln \psi + \mathcal{D}\frac{\Delta \psi}{\psi} \right\} = -\nabla \Phi/m. \tag{71}$$

Integrating this equation finally yields

$$\mathcal{D}^2 \Delta \psi + i\mathcal{D}\frac{\partial}{\partial t}\psi - \frac{\Phi}{2m}\psi = 0, \tag{72}$$

up to an arbitrary phase factor which may be set to zero by a suitable choice of the ψ phase.

In the case of standard quantum mechanics, as applied to microphysics, the necessary choice $\mathcal{S}_0 = 2m\mathcal{D}$ means that there is a natural link between the Compton relation and the Schrödinger equation. In this case, indeed, $\mathcal{S}_0$ is nothing but the fundamental action constant $\hbar$, while $\mathcal{D}$ defines the fractal/non-fractal transition (i.e., the transition from explicit scale dependence to scale independence in the rest frame), $\lambda = 2\mathcal{D}/c$. Therefore, the relation $\mathcal{S}_0 = 2m\mathcal{D}$ becomes a relation between mass and the fractal to scale-independence transition, which writes

$$\lambda_c = \frac{\hbar}{mc} \ . \tag{73}$$

We recognize here the definition of the Compton length. Its profound meaning - i.e., up to the fundamental constants $\hbar$ and c, that of inertial mass itself - is thus given, in our framework by the transition scale from explicit scale dependence (at small scales) to scale-independence (at large scales). We note that this length-scale is to be understood as a structure of scale space, not of standard space.

We recover, in this case, the standard form of Schrödinger's equation

$$\frac{\hbar^2}{2m}\Delta\psi + i\hbar\frac{\partial}{\partial t}\psi = \Phi\psi. \tag{74}$$

The statistical meaning of the wave function (Born postulate) can now be deduced from the very construction of the theory. Even in the case of only one particle, the virtual geodesic family is infinite (this remains true even in the zero particle case, i.e., for the vacuum field). The particle properties are assimilated to those of a random subset of the geodesics in the family, and its probability to be found at a given position must be proportional to the density of the geodesic fluid. This density can easily be calculated in our formalism, since the imaginary part of Eq. (72) writes in terms of $\rho = \psi\psi^\dagger$

$$\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho V) = 0, \tag{75}$$

where V is the real part of the complex velocity, which is identified, at the classical limit, with the classical velocity. This equation is recognized as the equation of continuity, implying that $\rho = \psi\psi^\dagger$ represents the fluid density which is proportional to the probability density, thus ensuring the validity of Born's postulate. The remarkable new feature that allows us to obtain such a result is that the continuity equation is not written as an additional a priori equation, but is now a part of our generalized equation of dynamics.

The von Neumann postulate is also easily recovered in such a geometric interpretation. Indeed, we may identify a measurement with a selection of the sub-sample of the geodesics family that keeps only the geodesics having the geometric properties corresponding to the measurement result. Therefore, just after the measurement, the "particle" is in the state given by the measurement.

These results have been generalized to the Klein-Gordon equation [17, 18] in the motion-relativistic case (by taking into account not only a fractal space, but also fractal time), then to the Dirac equation [23]: in this last case, one takes into account the symmetry breaking under the $(dx^\mu \leftrightarrow -dx^\mu)$ reflexion, that leads to introduce probability amplitudes described by complex quaternions, which are equivalent to Dirac bi-spinors. The Klein-Gordon equation is demonstrated to be valid also for these bi-quaternions, then to transform spontaneouly in the Dirac equation.

3 Application to gravitation

3.1 Curved and fractal space-time

Applications of the scale relativity theory to the problem of the formation and evolution of gravitational structures have been presented in several previous works [4, 18, 19]. We shall only briefly sum up here the principles and methods used in such an attempt, then quote some of the main results obtained.

In its present acceptance, gravitation is understood as the various manifestations of the geometry of space-time at large scales. Up to now, in the framework of Einstein's theory, this geometry was considered to be Riemannian. However, in the new framework of scale relativity, the geometry of space-time is assumed to be characterized not only by curvature, but also by fractality on some ranges of scale. As we shall see in what follows, fractality manifests itself, in the simplest case, in terms of the appearance of a new scalar field. We have suggested [41] that this new field is able to explain, without additional matter, the various astrophysical effects which have been, up to now, tentatively attributed to unseen "dark" matter.

Let us consider the motion of a free particle in a curved space-time whose spatial part is also fractal. One can define a motion+scale covariant derivative that combine the general-relativistic covariant derivative (which describes the effects of curvature) and the scale-relativistic covariant derivative (which describes the effects of fractality), namely,

$$\frac{\bar{D}A^\mu}{ds} = [\frac{\partial}{\partial s} + \mathcal{V}^\nu \partial_\nu + i\mathcal{D}\partial^\nu \partial_\nu] A^\mu + \Gamma^\mu_{\rho\nu} \mathcal{V}^\rho A^\nu. \tag{76}$$

The equation of motion of a free particle can now be written as a geodesics equation by using this covariant derivative. However, one should take care that the combination of the two covariant derivatives imply the appearance of a new term in the geodesics equation [29, 19, 30]. This is easily established by starting from Pissondes's quadratic invariant, [31] $\mathcal{V}_\mu \mathcal{V}^\mu + 2i\mathcal{D}\partial_\mu \mathcal{V}^\mu = 1$, which is a re-expression of Eq. (64) and becomes in the general-relativistic case:

$$\mathcal{V}_\mu \mathcal{V}^\mu + 2i\mathcal{D}\; D_\mu \mathcal{V}^\mu = 1, \tag{77}$$

where we now have $\mathcal{V}_\mu \mathcal{V}^\mu = g_{\mu\nu} \mathcal{V}^\mu \mathcal{V}^\nu$. The equations of motion are obtained by differentiating this relation. One obtains [30]:

$$\frac{d\!\!\!/}{ds} \mathcal{V}^\mu + \Gamma^\mu_{\nu\rho} \mathcal{V}^\nu \mathcal{V}^\rho - i\mathcal{D}\; R^\mu_\nu \mathcal{V}^\nu = 0. \tag{78}$$

This equation can be integrated in terms of a generalized "Einstein-Klein-Gordon" equation of motion that writes:

$$\frac{4\mathcal{D}^2}{c^2} [g_{\mu\nu}\, \partial^\mu \partial^\nu \psi + \partial_\nu (\ln \sqrt{-g})\, \partial^\nu \psi] = -1, \tag{79}$$

where g is the metrics determinant. A detailed study of this equation, although interesting, is outside the scope of the present contribution.

3.2 Gravitational Schrödinger equation

We shall consider in what follows only the Newtonian limit and situations where the additional term is null or negligible (for example, the Kepler problem). In this case the equation of motion is reduced to an equation that keeps the form of Newton's fundamental equation of dynamics in a gravitational field, namely,

$$\frac{\bar{D}\mathcal{V}}{dt} = \frac{d\mathcal{V}}{dt} + \nabla\left(\frac{\phi}{m}\right) = 0, \tag{80}$$

where ϕ is the Newtonian potential energy. As demonstrated hereabove, once written in terms of ψ, this equation can be integrated to yield a gravitational Newton-Schrödinger equation :

$$\mathcal{D}^2 \bigtriangleup \psi + i\mathcal{D}\frac{\partial}{\partial t}\psi = \frac{\phi}{2m}\,\psi. \tag{81}$$

Since the imaginary part of this equation is the equation of continuity, and basing ourselves on our description of the motion in terms of an infinite family of geodesics, $P = \psi\psi^\dagger$ can be interpreted as giving the probability density of the particle position.

Note however that the situation and therefore the interpretation are different here from the application of the theory to the microphysical domain. The two main differences are:

(i) while in the microscopic realm elementary "particles" can be defined as the geodesics themselves (their defining properties such as mass, spin or charge being defined as internal geometric properties, see [18, 23]), in the macroscopic realm there does exist actual particles that follow the geodesics;

(ii) while differentiability is definitively lost toward the small scales in the microphysical domain, the macroscopic quantum theory is valid only beyond some time-scale transition (and/or space-scale transition) which is an horizon of predictibility. Therefore in this last case there is an underlying classical theory, which means that the quantum macroscopic approach is a hidden variable theory [19].

Even though it takes this Schrödinger-like form, equation (81) is still in essence an equation of gravitation, so that it must keep the fundamental properties it owns in Newton's and Einstein's theories. Namely, it must agree with the equivalence principle [32, 33, 34], i.e., it is independent of the mass of the test-particle. In the Kepler central potential case ($\phi = -GMm/r$), GM provides the natural length-unit of the system under consideration. As a consequence, the parameter $\mathcal{D}$ takes the form:

$$\mathcal{D} = \frac{GM}{2w}, \tag{82}$$

where w is a fundamental constant that has the dimension of a velocity. The ratio $\alpha_g = w/c$ actually plays the role of a macroscopic gravitational coupling constant [34, 35]).

3.3 "Dark" potential

Let us now compare our approach with the standard theory of gravitational structure formation and evolution. Instead of the Euler-Newton equation and of the continuity equation which are used in the standard approach, we write the only above Newton-Schrödinger equation. In both cases, the Newton potential is given by the Poisson equation. Two situations can be considered: (i) when the 'orbitals', which are solutions of the motion equation, can be considered as filled with the particles (e.g., planetesimals in the case of planetary systems formation, interstellar gas and dust in the case of star formation, etc...), the mass density ρ is proportional to the probability density $P = \psi\psi^\dagger$: this situation is relevant in particular for addressing problems of structure formation; (ii) another possible situation concerns test bodies which are not in sufficiently large number to change the matter density, but whose motion is nevertheless submitted to the Newton-Schrödinger equation: this case is relevant for the anomalous dynamical effects which have up to now been attributed to unseen, "dark" matter.

By separating the real and imaginary parts of the Schrödinger equation we get respectively a generalized Euler-Newton equation (written here in terms of the Newtonian potential energy ϕ) and a continuity equation:

$$m\,(\frac{\partial}{\partial t} + V\cdot\nabla)V = -\nabla(\phi + Q), \tag{83}$$

$$\frac{\partial P}{\partial t} + \mathrm{div}(PV) = 0, \tag{84}$$

$$\Delta\phi = 4\pi G\rho\, m. \tag{85}$$

In the case $P \propto \rho$ this system of equations is equivalent to the classical one used in the standard approach of gravitational structure formation, except for the appearance of an extra potential energy term Q that writes:

$$Q = -2m\mathcal{D}^2\frac{\Delta\sqrt{P}}{\sqrt{P}}. \tag{86}$$

The existence of this potential energy, which has been identified as such by Bohm in the microphysical case (but without an understanding of its origin, since it was derived from the a priori axioms of quantum mechanics) is, in our approach, readily demonstrated and understood: namely, it is the very manifestation of the fractality of space [43], in similarity with Newton's potential being a manifestation of curvature.

In the case (i) where actual particles achieve the probability density distribution (structure formation), we have $\rho = \rho_0 P$; then the Poisson equation (i.e., the field equation) becomes $\Delta\phi = 4\pi G m\rho_0\psi\psi^\dagger$ and it is therefore strongly interconnected with the Schrödinger equation (i.e., the particle motion equation). An equation for matter alone can finally be written [19] (which has automatically its equivalent in an equation for the potential alone):

$$\Delta\left(\frac{\mathcal{D}^2\Delta\psi + i\mathcal{D}\partial\psi/\partial t}{\psi}\right) - 2\pi G\rho_0|\psi|^2 = 0. \tag{87}$$

This is a Hartree equation of the kind which is encountered in the description of superconductivity. That the self-structuring gravitational fluid may own superconducting properties has already been suggested by Agop et al. (see e.g.

[44] and references therein). We expect its solutions to provide us with general theoretical predictions for the structures (in position and velocity space) of self-gravitating systems at every scales [36]. This expectation is already supported by the agreement of several partial solutions with astrophysical observational data [4, 32, 35, 37, 38, 39, 40, 45].

Indeed, the theory has been able to predict in a quantitative way a large number of new effects in the domain of gravitational structures. Moreover, these predictions have been successfully checked in various systems on a large range of scales and in terms of a common fundamental gravitational coupling constant, $w_0 = c\,\alpha_g = 144.7 \pm 0.7$ km/s. New structures have been theoretically predicted, then checked by the observational data in a statistically significant way, for our solar system, including distances of planets [4, 37] and satellites [45], sungrazer comet perihelions [46], obliquities and inclinations of planets and satellites [39], exoplanets semi-major axes [32, 35] and eccentricities [47], including planets around pulsars (for which a high precision is reached) [32, 40], double stars [38], planetary nebula [48], binary galaxies [18, 49], our local group of galaxies [48], clusters of galaxies and large scale structures of the universe [38, 48].

In the case (ii) of isolated test particles, the density of matter ρ may be nearly zero while the probability density P does exist, but only as a virtual quantity that determines the potential Q, without being effectively achieved by matter. In this situation, even though there is no matter at the point considered (except the test particle that is assumed to have a very low contribution), the effects of the potential Q are real (since it is the result of the structure of the geodesics two-fluid). This situation is quite similar to the Newton potential in vacuum around a mass. We have therefore suggested [41] that this extra-potential may be responsible for the various dynamical and lensing effects which are usually attributed to unseen "dark matter". This interpretation is supported by the fact that, for a stationary solution of the gravitational Schrödinger equation, one gets the general relation:

$$\frac{\phi + Q}{m} = \frac{E}{m} = \text{cst}, \tag{88}$$

where E/m can take only quantized values (which are related to the fundamental gravitational coupling [34], $\alpha_g = w/c$).

This result can be applied, as an example, to the motion of bodies in the outer regions of spiral galaxies. In these regions there is practically no longer any visible matter, so that the Newtonian potential (in the absence of additional dark matter) is Keplerian. While the standard Newtonian theory predicts for the velocity of the halo bodies $v \propto \phi^{1/2}$, i.e. $v \propto r^{-1/2}$, we predict in our theory $v \propto |(\phi + Q)/m|^{1/2}$, i.e., $v =$ constant. More specifically, assuming that the gravitational Schrödinger equation is solved for the halo objects in terms of the fundamental level wave function, one finds $Q_{pred} = -\frac{GMm}{2r_B}\left(1 - \frac{2r_B}{r}\right)$, where $r_B = GM/w_0^2$. This is exactly the result which is systematically observed in spiral galaxies (i.e., flat rotation curves) and which has motivated the assumption of the existence of dark matter. In other words, we suggest that the effects tentatively attributed to unseen matter are simply the result of the geometry of space-time. In this proposal, space-time is not only curved but also fractal beyond some given relative time and space-scales. While the curvature manifests itself in terms of the Newton potential, fractality manifests itself in terms of the new scalar potential Q, and then finally in terms of the anomalous dynamics and lensing effects.

4 Conclusion and prospect

The present contribution has mainly focused on the detailed demonstration of the Schrödinger equation in a non-differentiable space-time, constrained by the principle of relativity (applied to scale transformations). We have also briefly considered some applications to the problem of formation and evolution of gravitational structures, including the question of "dark matter".

Other developments and generalizations of the theory, that have not been considered in the present contribution, have been presented elsewhere: e.g., special scale relativity [15], which involves the introduction of a fifth dimension that plays for scale laws a role similar to that played by time for motion laws (in this framework, the Planck length-time-scale becomes a minimal, impassable horizon, which implies several consequences in elementary particle physics, see [32, 41]; scale dynamics [20, 42]; motion-scale coupling which leads to a reinterpretation of gauge invariance [17, 50], and is a first step toward general scale-motion relativity; variable fractal dimension [18]; generalized Schrödinger equations in the case of the rotational motion of solids, of Euler and Navier-Stokes equations, of equations with dissipation and of scalar field equations (whose solutions give probability amplitudes for the potential) [19]; applications in cosmology [4, 18] (that lead to a theoretical prediction of the value of the cosmological constant, $\Omega_\Lambda = 0.7 \pm 0.15$ which is now supported by observational measurements).

Among the possible generalizations of the theory, one can also abandon the differentiability, not only in the usual space-time of positions and instants, but also in the space of scales itself. All the previous construction can again apply to this deeper level of description, that leads to the introduction of a "quantum mechanics of scale". In this framework, which is equivalent to a "third quantization", fractals "objects" of a new type can be defined. Let us recall indeed, that, up to now, various kinds of fractal structures have been physically defined:

(1) The first case is given by Mandelbrot's fractal objects, that own structures at well defined scales.

(2) The second case is given by scale-relativistic fractals (the ones that we consider throughout this work), for which only scale ratios have a physical meaning, never an absolute scale.

(3) A still more profound description of fractal structures allows the scale ratios to become themselves variable in terms of the space-time coordinates, i.e., to become a field. This is an important domain of development of the scale-

relativity theory, since it leads to a re-interpretation of the very nature of gauge transformations, and therefore of the gauge field and of their associated charges [17, 18, 50].

By giving up differentiability in the scale space, a still new case of fractal structures appears. These are "quantum fractals", which are defined by the probability for a scale ratio to have some given value, which is deduced from the solution of a Schrödinger equation acting in scale space. Such an approach may be particularly relevant for the description and understanding of complex systems (in particular in biology), characterized in particular by the occurrence of imbricated hierarchical levels of organisation and by evolution between these levels [51, 52, 53, 54, 55].

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