The Cauchy problem and numerical relativity

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Hommage à Yvonne Choquet-Bruhat
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Solving the Einstein equation

Once written in a coordinate system, the Einstein equation

$$\boldsymbol{G}(\boldsymbol{g}) = 8\pi \boldsymbol{T}$$

is turned into a rather involved system of 10 second-order partial differential equations (PDE) for the metric components $g_{\alpha\beta}$.

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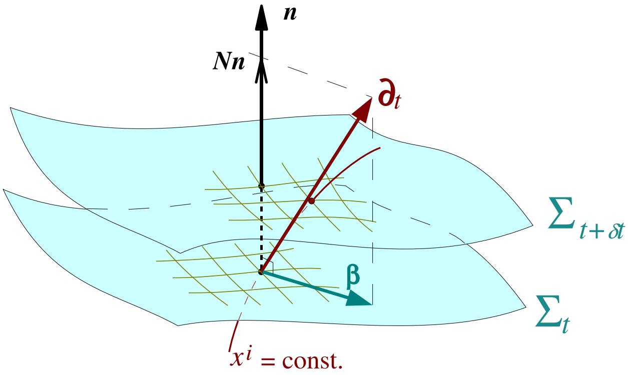
Formulation of the Cauchy problem in a geometric framework: 3+1 formalism

3+1 foliation of spacetime



Spacetime $(\mathscr{M}, \boldsymbol{g})$ assumed to be **globally hyperbolic**: $\exists$ a **foliation** (or **slicing**) of the spacetime manifold $\mathscr{M}$ by spacelike hypersurfaces Σ_t :

$$\mathscr{M} = \bigcup_{t\in\mathbb{R}} \Sigma_t$$

$\boldsymbol{n}$: unit normal to Σ_t : $n^\alpha = -N\nabla^\alpha t$

N : lapse function

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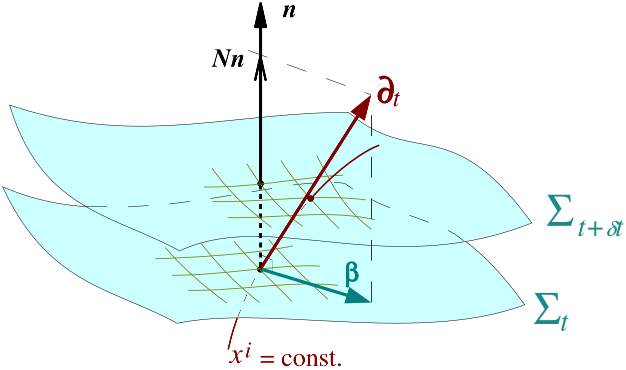

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On each Σ_t, coord. $(x^i) = (x^1, x^2, x^3)$
$\Longrightarrow$ $(x^\alpha) = (t, x^1, x^2, x^3)$ coord. on $\mathscr{M}$

$\boldsymbol{\partial}_t = N\boldsymbol{n} + \boldsymbol{\beta}$ with $\boldsymbol{n}\cdot\boldsymbol{\beta} = 0$
$\Longrightarrow$ shift vector $\boldsymbol{\beta}$

Induced metric (first fundamental form)

The **induced metric** or **first fundamental form** on Σ_t is the bilinear form $\boldsymbol{\gamma}$ defined by

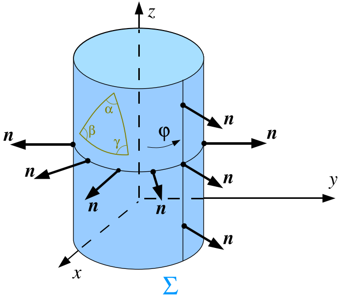
$$\forall(\boldsymbol{u},\boldsymbol{v}) \in T_p\Sigma_t \times T_p\Sigma_t, \quad \boldsymbol{\gamma}(\boldsymbol{u},\boldsymbol{v}) := \boldsymbol{g}(\boldsymbol{u},\boldsymbol{v})$$

Σ_t spacelike $\iff$ $\boldsymbol{\gamma}$ positive definite (Riemannian metric)

Spacetime metric tensor in terms of the lapse, shift and induced metric:

$$\boxed{g_{\mu\nu}\, dx^\mu\, dx^\nu = -N^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)}$$

Extrinsic curvature (second fundamental form)



The **extrinsic curvature** (or **second fundamental form**) of Σ_t is a bilinear form field $\boldsymbol{K}$ on Σ_t describing the "bending" of Σ_t in $(\mathscr{M}, \boldsymbol{g})$, i.e. the variation of the normal vector $\boldsymbol{n}$ along Σ_t:

$$\forall (\boldsymbol{u}, \boldsymbol{v}) \in T_p\Sigma_t \times T_p\Sigma_t, \quad \boldsymbol{K}(\boldsymbol{u}, \boldsymbol{v}) := -\boldsymbol{u} \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} \boldsymbol{n}$$

Weingarten property: $\boldsymbol{K}$ is symmetric: $\boldsymbol{K}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{K}(\boldsymbol{v}, \boldsymbol{u})$

Trace: $K := \mathrm{tr}_{\boldsymbol{\gamma}} \boldsymbol{K} = \gamma^{ij} K_{ij}$ = (3 times) the **mean curvature** of Σ_t

Σ_t part of a foliation $\Longrightarrow$ alternative expression of $\boldsymbol{K}$:

$$\boxed{\boldsymbol{K} = -\frac{1}{2} \boldsymbol{\mathcal{L}}_{\boldsymbol{n}} \boldsymbol{\gamma}}$$

Example (Cylinder)

$\boldsymbol{\gamma} = \mathbf{d}z^2 + R^2 \mathbf{d}\varphi^2$

- $\mathrm{Riem}(\boldsymbol{\gamma}) = 0$ (flat metric)
- $\boldsymbol{K} \neq 0$

3+1 Einstein system

Thanks to the Gauss, Codazzi and Ricci equations, the Einstein equation

$$ {}^4R_{\alpha\beta} - \frac{1}{2} {}^4R \, g_{\alpha\beta} = 8\pi T_{\alpha\beta} $$

is equivalent to the system

- $\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}} \right) \gamma_{ij} = -2NK_{ij}$ *kinematical relation* $\boldsymbol{K} = -\frac{1}{2} \boldsymbol{\mathcal{L}}_{\boldsymbol{n}} \boldsymbol{\gamma}$
- $\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}} \right) K_{ij} = -D_i D_j N + N \left\{ R_{ij} + K K_{ij} - 2K_{ik} K^k{}_j + 4\pi \left[(S - E)\gamma_{ij} - 2S_{ij} \right] \right\}$
- $R + K^2 - K_{ij}K^{ij} = 16\pi E$ *Hamiltonian constraint*
- $D_j K^j{}_i - D_i K = 8\pi p_i$ *momentum constraint*

$$ T_{\alpha\beta} = S_{\alpha\beta} + n_\alpha p_\beta + p_\alpha n_\beta + E n_\alpha n_\beta $$

The full PDE system

Supplementary equations:

$$
\begin{aligned}
& D_i D_j N = \frac{\partial^2 N}{\partial x^i \partial x^j} - \Gamma^k_{\ ij} \frac{\partial N}{\partial x^k} \\
& D_j K^j_{\ i} = \frac{\partial K^j_{\ i}}{\partial x^j} + \Gamma^j_{\ jk} K^k_{\ i} - \Gamma^k_{\ ji} K^j_{\ k} \\
& \mathcal{L}_{\boldsymbol{\beta}}\, \gamma_{ij} = \frac{\partial \beta_i}{\partial x^j} + \frac{\partial \beta_j}{\partial x^i} - 2\Gamma^k_{\ ij} \beta_k \\
& \mathcal{L}_{\boldsymbol{\beta}}\, K_{ij} = \beta^k \frac{\partial K_{ij}}{\partial x^k} + K_{kj} \frac{\partial \beta^k}{\partial x^i} + K_{ik} \frac{\partial \beta^k}{\partial x^j} \\
& R_{ij} = \frac{\partial \Gamma^k_{\ ij}}{\partial x^k} - \frac{\partial \Gamma^k_{\ ik}}{\partial x^j} + \Gamma^k_{\ ij} \Gamma^l_{\ kl} - \Gamma^l_{\ ik} \Gamma^k_{\ lj} \\
& R = \gamma^{ij} R_{ij} \\
& \Gamma^k_{\ ij} = \frac{1}{2} \gamma^{kl} \left(\frac{\partial \gamma_{lj}}{\partial x^i} + \frac{\partial \gamma_{il}}{\partial x^j} - \frac{\partial \gamma_{ij}}{\partial x^l} \right)
\end{aligned}
$$

History of the 3+1 Einstein equations

- G. Darmois (1927): 3+1 Einstein equations in terms of (γ_{ij}, K_{ij}) with $N = 1$ and $\boldsymbol{\beta} = 0$ (Gaussian normal coordinates)
- A. Lichnerowicz (1939) : $N \neq 1$ and $\boldsymbol{\beta} = 0$ (normal coordinates)
- Y. Choquet-Bruhat (1956) : $N \neq 1$ and $\boldsymbol{\beta} \neq 0$ (general coordinates in Cartan's moving frame)
- R. Arnowitt, S. Deser & C.W. Misner (1962) : *Hamiltonian formulation* of GR based on a 3+1 decomposition in terms of (γ_{ij}, π^{ij})
  NB: spatial projection of *Einstein tensor* instead of *Ricci tensor* in previous works
- J. Wheeler (1964) : coined the terms *lapse* for N and *shift* for $\boldsymbol{\beta}$
- J.W. York (1979) : modern 3+1 decomposition based on spatial projection of *Ricci tensor*

The Cauchy problem

The first two equations of the 3+1 Einstein system can be recast as

$$\frac{\partial^2 \gamma_{ij}}{\partial t^2} = F_{ij}\left(\gamma_{kl}, \frac{\partial \gamma_{kl}}{\partial x^m}, \frac{\partial \gamma_{kl}}{\partial t}, \frac{\partial^2 \gamma_{kl}}{\partial x^m \partial x^n}\right) \tag{1}$$

allowing to formulate a **Cauchy problem:** given initial data at $t = 0$: γ_{ij} and $\frac{\partial \gamma_{ij}}{\partial t}$, find a solution for $t > 0$

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allowing to formulate a **Cauchy problem:** given initial data at $t=0$: γ_{ij} and $\frac{\partial \gamma_{ij}}{\partial t}$, find a solution for $t>0$

This Cauchy problem is subject to the constraints

- $R + K^2 - K_{ij}K^{ij} = 16\pi E$ *(Hamiltonian constraint)*
- $D_j K^j_{\ \ i} - D_i K = 8\pi p_i$ *(momentum constraint)*

Preservation of the constraints

Thanks to the Bianchi identities, it can be shown that if the constraints are satisfied at $t=0$, they are preserved by the evolution system (1), *provided that* $\nabla_\beta T^{\alpha\beta} = 0$ *is maintained*

Existence and uniqueness of solutions

Question:

Given a set $(\boldsymbol{\gamma}, \boldsymbol{K}, E, \boldsymbol{p})$ on a 3-dimensional manifold Σ_0, where

- $\boldsymbol{\gamma}$ a Riemannian metric on Σ_0,
- $\boldsymbol{K}$ a symmetric bilinear form field on Σ_0,
- E and $\boldsymbol{p}$ are a scalar field and a 1-form on Σ_0 ,

which fulfill the constraint equations, does there exist a spacetime $(\mathscr{M}, \boldsymbol{g}, \boldsymbol{T})$ such that $(\boldsymbol{g}, \boldsymbol{T})$ fulfills the Einstein equation and Σ_0 can be embedded as a hypersurface of $\mathscr{M}$ with induced metric γ and extrinsic curvature $\boldsymbol{K}$?

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Answer:

- the solution exists and is unique in a vicinity of Σ_0 for **real analytic** initial data (Cauchy-Kovalevskaya theorem) [Darmois (1927)], [Lichnerowicz (1939)]
- the solution exists and is unique in a vicinity of Σ_0 for **generic** (i.e. smooth) initial data [Choquet-Bruhat (1952)]
- there exists a unique maximal solution [Choquet-Bruhat & Geroch (1969)]

Choquet-Bruhat's general theorem for non-linear PDEs (1952)

Consider a system of n non-linear second-order PDE's of the type

$$A^{\mu\nu}\frac{\partial^2 W_s}{\partial x^\mu \partial x^\nu} + Q_s = 0 \qquad (2)$$

for n unknown functions $W_s = W_s(x^0 = t, x^1, x^2, x^3)$, $1 \le s \le n$, where $A^{\mu\nu} = A^{\mu\nu}(x^\alpha, W_s)$ and $Q_s = Q_s(x^\alpha, W_s, \partial_\alpha W_s)$, along with the initial conditions

$$W_s|_{t=0} = \Phi_s(x^1, x^2, x^3) \quad \text{and} \quad \left.\frac{\partial W_s}{\partial t}\right|_{t=0} = \Psi_s(x^1, x^2, x^3) \qquad (3)$$

where Φ_s (resp. Ψ_s) is of class C^5 (resp. C^4) with Lipschitz continuous derivatives.
Suppose that, in the vicinity[a] of $t = 0$,

- $A^{\mu\nu}(x^\alpha, W_s)$ and $Q_s(x^\alpha, W_s, \partial_\alpha W_s)$ are of class C^4 with Lipschitz continuous derivatives;
- $A^{00} > 0$ and A^{ij} is negative definite (system (2) is *hyperbolic*).

Then there exists a unique solution $(W_s)_{1\le s\le n}$ to (2)-(3) in the vicinity of $t = 0$, which is of class C^4.

---

[a] See [Choquet-Bruhat (1952)] for the precise conditions

Application to general relativity

Vacuum Einstein equation:

$$\boxed{{}^4R_{\alpha\beta} = 0}$$

with

$${}^4R_{\alpha\beta} = \underbrace{-\frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta} + \frac{1}{2} g_{\alpha\sigma} \partial_\beta \Gamma^\sigma + \frac{1}{2} g_{\beta\sigma} \partial_\alpha \Gamma^\sigma}_{\text{2nd order derivatives of } \boldsymbol{g}} + \underbrace{Q_{\alpha\beta}(g_{\mu\nu}, \partial_\sigma g_{\mu\nu})}_{\text{0th and 1st der.}}$$

$$\Gamma^\alpha := g^{\mu\nu}\, {}^4\Gamma^\alpha_{\ \ \mu\nu} = -\nabla_\mu \nabla^\mu x^\alpha = -\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\alpha} \right)$$

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In harmonic coordinates $\boxed{\Gamma^\alpha = 0}$, the vacuum Einstein equation reduces to

$$\mathcal{R}^{\mathrm{H}}_{\alpha\beta} := -\frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta} + Q_{\alpha\beta}(g_{\mu\nu}, \partial_\sigma g_{\mu\nu}) = 0 \qquad (4)$$

which is of the type (2) if the hypersurface $t = 0$ is spacelike ($g^{00} < 0$):
$A^{\mu\nu} = -\frac{1}{2} g^{\mu\nu}$, $W_s = g_{\alpha\beta}$, $s = (\alpha, \beta)$, $n = 10$.

Choquet-Bruhat's strategy for the Cauchy problem of general relativity

Assume that the constraint equations are fulfilled on the hypersurface $t = 0$: they are equivalent to

$$\left. {}^4G^0{}_\alpha \right|_{t=0} = 0 \qquad (5)$$

More precisely: ${}^4G^0{}_0 = -\frac{1}{2}(R + K^2 - K_{ij}K^{ij}) + \frac{\beta^i}{N}(D_jK^j{}_i - D_iK)$ and ${}^4G^0{}_i = \frac{1}{N}(D_jK^j{}_i - D_iK)$

Without any loss of generality choose coordinates such that the harmonicity condition holds on Σ_0:

$$\left. \Gamma^\alpha \right|_{t=0} = 0 \qquad (6)$$

1. Apply the general PDE theorem to get a solution $g_{\alpha\beta}$ to $\mathcal{R}^{\mathrm{H}}_{\alpha\beta} = 0$ [Eq. (4)] in the vicinity of Σ_0. The Ricci and Einstein tensors of the solution are then

$$ {}^4R_{\alpha\beta} = \frac{1}{2}\left(g_{\alpha\sigma}\partial_\beta\Gamma^\sigma + g_{\beta\sigma}\partial_\alpha\Gamma^\sigma\right)$$

$$ {}^4G_{\alpha\beta} = \frac{1}{2}\left(g_{\alpha\sigma}\partial_\beta\Gamma^\sigma + g_{\beta\sigma}\partial_\alpha\Gamma^\sigma - \partial_\sigma\Gamma^\sigma g_{\alpha\beta}\right) \qquad (7)$$

2. (6) implies $\left.\partial_i\Gamma^\alpha\right|_{t=0} = 0$, so that (5) and (7) lead to

$$\left.\partial_t\Gamma^\alpha\right|_{t=0} = 0 \qquad (8)$$

Choquet-Bruhat's strategy for the Cauchy problem of general relativity

3. The Bianchi identity $\nabla^\mu G_{\mu\alpha} = 0$ along with (7) leads to

$$g^{\mu\nu}\partial_\mu\partial_\nu\Gamma^\alpha + P^\alpha(x^\beta, \partial_\mu\Gamma^\nu) = 0 \tag{9}$$

4. Apply the general PDE theorem to the system (9) with the initial conditions (6) and (8) to conclude that the unique solution of (9) in the vicinity of Σ_0 is $\boxed{\Gamma^\alpha = 0}$.
5. It follows that the obtained solution $g_{\alpha\beta}$ of $\mathcal{R}^{\rm H}_{\alpha\beta} = 0$ is the solution of the vacuum Einstein equation ${}^4R_{\alpha\beta} = 0$ expressed in harmonic coordinates.

Generalized harmonic evolution system

The Choquet-Bruhat result can be extended from $\Gamma^\alpha = 0$ to

$$\Gamma^\alpha = -H^\alpha(x^\beta, g_{\beta\gamma})$$

where the H^α's are arbitrary prescribed algebraic functions [Friedrich, CMP **100**, 525 (1985)].
Indeed, since H^α does not depend upon $\partial_\alpha g_{\beta\gamma}$, the vacuum Einstein equation ${}^4R_{\alpha\beta} = 0$ still takes the hyperbolic form (4).

By choosing H^α, one can accommodate for any coordinate system.

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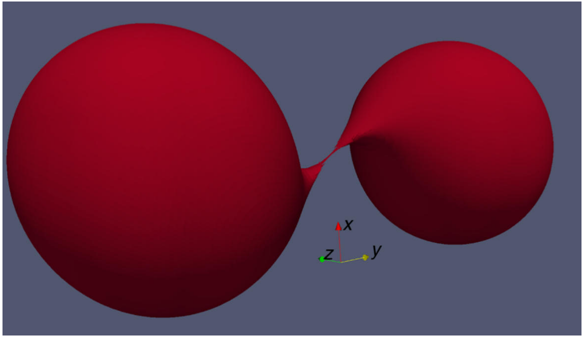
First use in numerical relativity for the binary black hole problem by F. Pretorius (2005) [CQG **22**, 425].

Generalized harmonic formulation can be tuned to damp out constraint-violating modes during numerical evolution [Lindblom, Scheel, Kidder, Owen & Rinne, CQG **23**, S447 (2006)].

Generalized harmonic evolution system



Event horizon in a binary black hole merger

[Cohen et al., PRD **85**, 024031 (2012)]

Generalized harmonic scheme used in the Spectral Einstein Code (SpEC) (L. Kidder, L. Lindblom, H. Pfeiffer, M. Scheel et al.)

`https://www.black-holes.org/for-researchers/spec`

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