Dynamics of strings and branes, with application to junctions and vortons

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Contents :

1 Worldsheet curvature analysis

1-1 The first fundamental tensor

1-2 The inner and outer curvature tensors

1-3 The second fundamental tensor

1-4 The extrinsic curvature vector and the conformation tensor

1-5 Codazzi, Gauss, and Schouten identities

1-6 Internal Ricci and Conformal Curvatures

1-7 Special case of string worldsheet

1.1 The first fundamental tensor

The development of geometrical intuition and of computationally efficient methods for use in string and membrane theory has been hampered by a tradition of publishing results in untidy, highly gauge dependent, notation (one of the causes being the undue influence still exercised by Eisenhart's obsolete treatise "Riemannian Geometry" [1]). For the intermediate steps in particular calculations it is of course frequently useful and often indispensible to introduce specifically adapted auxiliary structures, such as curvilinear worldsheet coordinates σ^i ($i = 0$, ..., d- 1) and the associated bitensorial derivatives, namely

$$x^\mu{}_{,i} = \frac{\partial x^\mu}{\partial \sigma^i} \, . \tag{1}$$

It is also frequently useful to introduce (adapted) orthonormal frame vectors, namely an internal subset of vectors $\iota_A{}^\mu$ ($A = 0, \ldots, \mathrm{d}{-}1$) tangential to the worldsheet and an external subset of vectors $\lambda_X{}^\mu$ ($X = 1, \ldots, \mathrm{n}{-}\mathrm{d}$) orthogonal to the worldsheet, as characterised by

$$\iota_A{}^\mu \iota_{B\mu} = \eta_{AB} \,, \qquad \iota_A{}^\mu \lambda_{X\mu} = 0 \,, \qquad \lambda_X{}^\mu \lambda_{Y\mu} = \delta_{XY} \,, \qquad (2)$$

where η_{AB} is a fixed d-dimensional Minkowski metric and the Kronecker matrix δ_{XY} is a fixed ($\mathrm{n}{-}\mathrm{d}$)-dimensional Cartesion metric, but even in the most recent literature there are still many examples of insufficient effort to sort out the ensuing clutter of indices of different kinds (Greek or Latin, early or late, small or capital) by grouping the various contributions into simple tensorially covariant combinations.

Another inconvenient feature of many publications is that results have been left in a form that depends on some particular gauge choice (such as the conformal gauge for internal string coordinates) which obscures the relationship with other results concerning the same system but in a different gauge.

The strategy adopted here [2] aims at minimising such problems (they can never be entirely eliminated) by working as far as possible with a single kind of tensor index, which must of course be the one that is most fundamental, namely that of the n-dimensional coordinates, x^μ, on the background spacetime with metric $g_{\mu\nu}$.

Thus, to avoid dependence on the internal frame index A (which is lowered and raised by contraction with the fixed d-dimensional Minkowski metric η_{AB} and its inverse η^{AB}) and on the external frame index X (which is lowered and raised by contraction with the fixed (n-d)-dimensional Cartesian metric δ_{XY} and its inverse δ^{XY}), the separate internal frame vectors $\iota_A{}^\mu$ and external frame vectors $\lambda_X{}^\mu$ will as far as possible be eliminated in favour of the frame gauge independent combinations

$$\eta^\mu{}_\nu = \iota_A{}^\mu \iota^A{}_\nu \,, \qquad \perp^\mu_\nu = \lambda_X{}^\mu \lambda^X{}_\nu \,. \tag{3}$$

The former, $\eta^\mu{}_\nu$, is what will be called the (first) *fundamental tensor* of the metric, which acts as the (rank d) operator of tangential projection onto the world sheet, while the latter, $\perp^\mu_\nu$, is the (rank n–d) operator of projection orthogonal to the world sheet.

The same principle applies to the avoidance of unnecessary involvement of the internal coordinate indices which are lowered and raised by contraction with the induced metric on the worldsheet as given by

$$\eta_{ij} = g_{\mu\nu} x^{\mu}{}_{,i} x^{\nu}{}_{,j} \,, \tag{4}$$

and with its contravariant inverse η^{ij}. After being cast (by index raising if necessary) into its contravariant form, any internal coordinate tensor can be directly projected onto a corresponding background tensor in the manner exemplified by the intrinsic metric itself, which gives

$$\eta^{\mu\nu} = \eta^{ij} x^{\mu}{}_{,i} x^{\nu}{}_{,j} \,. \tag{5}$$

The formula (5) provides an alternative (more direct) prescription for the fundamental tensor that was previously introduced via the use of the internal frame in (3). This approach also provides a direct prescription for the complementary orthogonal projector that was introduced via an external frame in (3) but is also obtainable immediately from (5) as

$$\perp^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} - \eta^{\mu}{}_{\nu} \,. \tag{6}$$

As well as having the separate operator properties

$$\eta^{\mu}{}_{\rho}\,\eta^{\rho}{}_{\nu} = \eta^{\mu}{}_{\nu} \;, \qquad\qquad \perp^{\mu}{}_{\rho}\perp^{\rho}{}_{\nu} = \perp^{\mu}{}_{\nu} \tag{7}$$

the tensors defined by (5) and (6) will evidently be related by

$$\eta^{\mu}{}_{\rho}\perp^{\rho}{}_{\nu} \;=\; 0 \;=\; \perp^{\mu}{}_{\rho}\eta^{\rho}{}_{\nu} \,. \tag{8}$$

1.2 The inner and outer curvature tensors

In so far as we are concerned with tensor fields such as the frame vectors whose support is confined to the d-dimensional world sheet, the effect of Riemannian covariant differentation ∇_μ along an arbitrary directions on the background spacetime will not be well defined, only the corresponding tangentially projected differentiation operation

$$\overline{\nabla}_\mu \overset{\text{def}}{=} \eta^\nu{}_\mu \nabla_\nu \,, \tag{9}$$

being meaningful for them, as for instance in the case of a scalar field φ for which the tangentially projected gradient is given in terms of internal coordinate differentiation simply by $\overline{\nabla}^\mu \varphi = \eta^{ij} x^\mu{}_{,i}\, \varphi_{,ji}$.

An irreducible basis for the various possible covariant derivatives of the frame vectors consists of the *internal rotation* pseudo-tensor $\rho_\mu{}^\nu{}_\rho$ and the *external rotation* (or "twist") pseudo-tensor $\varpi_\mu{}^\nu{}_\rho$ as given by

$$\rho_\mu{}^\nu{}_\rho = \eta^\nu{}_\sigma\, \iota^A{}_\rho \overline{\nabla}_\mu\, \iota_A{}^\sigma = -\rho_{\mu\rho}{}^\nu\,,$$

$$\varpi_\mu{}^\nu{}_\rho = \perp^\nu_\sigma\, \lambda^X{}_\rho \overline{\nabla}_\mu \lambda_X{}^\sigma = -\varpi_{\mu\rho}{}^\nu, \qquad (10)$$

together with their *mixed* analogue $K_{\mu\nu}{}^\rho$ which is obtainable in a pair of equivalent alternative forms given by

$$K_{\mu\nu}{}^\rho = \perp^\rho_\sigma\, \iota^A{}_\nu \overline{\nabla}_\mu\, \iota_A{}^\sigma = -\eta^\sigma_\nu\, \lambda_X{}^\rho \overline{\nabla}_\mu \lambda^X{}_\sigma\,. \qquad (11)$$

The reason for qualifying the fields (10) as "pseudo" tensors is that although they are tensorial in the ordinary sense with respect to changes of the background coordinates x^μ they are not geometrically well defined just by the geometry of the world sheet but are gauge dependent in the sense of being functions of the choice of the internal and external frames ${\iota_A}^\mu$ and ${\lambda_X}^\mu$. The gauge dependence of ${\rho_\mu}^\nu{}_\rho$ and ${\varpi_\mu}^\nu{}_\rho$ means that both of them can be set to zero at any chosen point on the worldsheet by choice of the relevant frames in its vicinity. However the condition for it to be possible to set these pseudo-tensors to zero throughout an open neigbourhood is the vanishing of the curvatures of the corresponding frame bundles as characterised with respect to the respective invariance subgroups SO(1,d–1) and SO(n–d) into which the full Lorentz invariance group SO(1,n–1) is broken by the specification of the d-dimensional world sheet orientation.

The existence of a gauge in which ${\rho_\mu}^\nu{}_\rho$ vanishes locally depends on the vanishing of the *inner curvature* of Riemannian type, that is obtainable (by a calculation of the type originally developed by Cartan that was made familiar to physicists by Yang Mills theory) as [3]

$$R_{\kappa\lambda}{}^\mu{}_\nu = 2\eta^\mu{}_\sigma\eta^\tau{}_\mu\eta^\pi{}_{[\lambda}\overline{\nabla}_{\kappa]}\rho_\pi{}^\sigma{}_\tau + 2\rho_{[\kappa}{}^{\mu\pi}\rho_{\lambda]\pi\nu}\,. \qquad (12)$$

The analogue for the “twist” tensor $\varpi_\mu{}^\nu{}_\rho$ the *outer curvature* of less familiar type that is given [3] by

$$\Omega_{\kappa\lambda}{}^\mu{}_\nu = 2\perp^\mu_\sigma\perp^\tau_\mu\eta^\pi{}_{[\lambda}\overline{\nabla}_{\kappa]}\varpi_\pi{}^\sigma{}_\tau + 2\varpi_{[\kappa}{}^{\mu\pi}\varpi_{\lambda]\pi\nu}\,. \qquad (13)$$

The frame gauge invariance of the expressions (12) and (13) – whereby $R_{\kappa\lambda}{}^\mu{}_\nu$ and $\Omega_{\kappa\lambda}{}^\mu{}_\nu$ are unambiguously well defined as tensors in the strictest sense of the word – is not obvious from the foregoing formulae, but will be made manifest in the alternative expressions given below.

1.3 The second fundamental tensor

Another, even more important, gauge invariance property that is not obvious from the traditional approach – as recapitulated above – is that of the entity $K_{\mu\nu}{}^{\rho}$ defined by the mixed analogue (11) of (10), which (unlike $\rho_{\mu}{}^{\nu}{}_{\rho}$ and $\varpi_{\mu}{}^{\nu}{}_{\rho}$, but like $R_{\kappa\lambda}{}^{\mu}{}_{\nu}$ and $\Omega_{\kappa\lambda}{}^{\mu}{}_{\nu}$) is a geometrically well defined tensor in the strict sense. The frame gauge independence of (11) can be seen from its agreement with the alternative – manifestly gauge independent – definition [4]

$$K_{\mu\nu}{}^{\rho} \overset{\text{def}}{=} \eta^{\sigma}{}_{\nu}\overline{\nabla}_{\mu}\eta^{\rho}{}_{\sigma}\,. \tag{14}$$

whereby the entity that we refer to as the *second fundamental tensor* is constructed directly from the the first fundamental tensor $\eta^{\mu\nu}$ as given by (5).

As this second fundamental tensor, $K_{\mu\nu}{}^{\rho}$ will play an important role in the work that follows, it is worth lingering over its essential properties. The expression (14) could of course be meaningfully applied not only to the fundamental projection tensor of a d-surface, but also to any (smooth) field of rank-d projection operators $\eta^{\mu}{}_{\nu}$ as specified by a field of arbitrarily orientated d-surface elements. What distinguishes the integrable case – in which the elements mesh together to form a well defined d-surface through the point under consideration – is the $Weingarten\ identity$, whereby that the tensor defined by (14) will have the symmetry property

$$K_{[\mu\nu]}{}^{\rho} = 0 \,, \tag{15}$$

an integrability condition that is derivable [4], [3] as a version of the well known Frobenius theorem.

As well as being symmetric, the tensor $K_{\mu\nu}{}^{\rho}$ is obviously tangential on the first two indices and also orthogonal on the last :

$$\perp^{\sigma}_{\mu} K_{\sigma\nu}{}^{\rho} = K_{\mu\nu}{}^{\sigma} \eta_{\sigma}{}^{\rho} = 0 \,. \tag{16}$$

This second fundamental tensor fully determines the tangential derivatives of the first fundamental tensor $\eta^{\mu}{}_{\nu}$ by the formula

$$\overline{\nabla}_{\mu} \eta_{\nu\rho} = 2 K_{\mu(\nu\rho)} \,, \tag{17}$$

(using round brackets to denote symmetrisation) and it can be seen to be characterisable by the condition that the orthogonal projection of the acceleration of any tangential unit vector field u^{μ} will be given by

$$u^{\mu} u^{\nu} K_{\mu\nu}{}^{\rho} = \perp^{\rho}_{\mu} \dot{u}^{\mu} \,, \qquad \dot{u}^{\mu} = u^{\nu} \nabla_{\nu} u^{\mu} \,. \tag{18}$$

1.4 The extrinsic curvature vector and the conformation tensor

It is very practical for a great many purposes to introduce the *extrinsic curvature vector* K^μ , defined as the trace of the second fundamental tensor, which is automatically orthogonal to the worldsheet,

$$K^\mu \overset{\text{def}}{=} K^\nu{}_\nu{}^\mu = \overline{\nabla}_\mu \eta^{\mu\nu} \,, \qquad \eta^\mu{}_\nu K^\nu = 0\,. \qquad (19)$$

It is useful for many specific purposes to work this out in terms of the intrinsic metric η_{ij} and its determinant $|\eta|$. For the tangentially projected gradient of a scalar field φ on the worldsheet, it suffices to use the simple expression

$$\overline{\nabla}^\mu \varphi = \eta^{ij} x^\mu{}_{,i} \varphi_{,j}\,.$$

However for a tensorial field (unless one is using Minkowski coordinates in a flat spacetime) the gradient will also have contributions involving the background Riemann Christoffel connection

$$\Gamma_{\mu\ \rho}^{\ \nu} = g^{\nu\sigma}\Big(g_{\sigma(\mu,\rho)} - \tfrac{1}{2} g_{\mu\rho,\sigma}\Big)\,. \tag{20}$$

The curvature vector is thus obtained in explicit detail as

$$K^\nu = \frac{1}{\sqrt{\|\eta\|}}\Big(\sqrt{\|\eta\|}\,\eta^{ij} x^\nu_{\ ,i}\Big)_{,j} + \eta^{ij} x^\mu_{\ ,i} x^\rho_{\ ,j} \Gamma_{\mu\ \rho}^{\ \nu}\,. \tag{21}$$

This expression is useful for specific computational purposes, but much of the literature on cosmic string dynamics has been made unnecessarily heavy by a tradition of working all the time with long strings of non tensorial terms such as those on the right of (21) rather than exploiting more succinct tensorial expressions, such as $K^\nu = \overline{\nabla}_\mu \eta^{\mu\nu}$.

As an alternative to the universally applicable tensorial approach advocated here, there is of course another more commonly used method of achieving succinctness in particular circumstances, which is to sacrifice gauge covariance by using specialised kinds of coordinate system.

In particular, for the case of a string, i.e. for a 2-dimensional worldsheet, it is standard practise to use conformal coordinates σ^0 and σ^1 so that the corresponding tangent vectors $\dot{x}^\mu = x^\mu_{\ ,0}$ and $x'^\mu = x^\mu_{\ ,1}$ satisfy the restrictions $\dot{x}^\mu x'_{\ \mu} = 0\,,\ \dot{x}^\mu \dot{x}_\mu + x'^\mu x'_{\ \mu} = 0\,,$ which implies $\sqrt{\|\eta\|} = x'^\mu x'_{\ \mu} = -\dot{x}^\mu \dot{x}_\mu\,,$ so that (21) simply gives

$$\sqrt{\|\eta\|}\, K^\nu = \ x''^\nu - \ddot{x}^\nu + (x'^\mu x'^\rho - \dot{x}^\mu \dot{x}^\rho)\Gamma_{\mu\ \rho}^{\ \nu}\,.$$

The physical specification of the extrinsic curvature vector (19) for a timelike d-surface in a dynamic theory provides what can be taken as the equations of extrinsic motion of the d-surface [4], [5], the simplest possibility being the "harmonic" condition $K^\mu = 0$ that is obtained (as will be shown in the following sections) from a surface measure variational principle such as that of the Dirac membrane model [6], or of the Goto-Nambu string model [7] whose dynamic equations in a flat background are therefore expressible with respect to a standard conformal gauge in the familiar form $x''^\mu - \ddot{x}^\mu = 0\,,$

There is a certain analogy between the Einstein vacuum equations, which impose the vanishing of the trace $\mathcal{R}_{\mu\nu}$ of the background spacetime curvature $\mathcal{R}_{\lambda\mu}{}^\rho{}_\nu\,,$ and the Dirac-Gotu-Nambu equations, which impose the vanishing of the trace K^ν of the second fundamental tensor $K_{\lambda\mu}{}^\nu\,,$

Just as it is useful to separate out the Weyl tensor [8], i.e. the trace free part of the Ricci background curvature which is the only part that remains when the Einstein vacuum equations are satisfied, so also analogously, it is useful to separate out the the trace free part of the second fundamental tensor, namely the extrinsic conformation tensor [3], which is the only part that remains when equations of motion of the Dirac - Goto - Nambu type are satisfied. Explicitly, the trace free *extrinsic conformation* tensor $\mathcal{C}_{\mu\nu}{}^{\rho}$ of a d-dimensional imbedding is defined [3] in terms of the corresponding first and second fundamental tensors $\eta_{\mu\nu}$ and $K_{\mu\nu}{}^{\rho}$ as

$$\mathcal{C}_{\mu\nu}{}^{\rho} \overset{\text{def}}{=} K_{\mu\nu}{}^{\rho} - \frac{1}{\mathrm{d}}\eta_{\mu\nu}K^{\rho} \ , \qquad \mathcal{C}^{\nu}{}_{\nu}{}^{\mu} = 0 \ . \qquad (22)$$

Like the Weyl tensor $\mathcal{W}_{\lambda\mu}{}^{\rho}{}_{\nu}$ of the background metric (whose definition is given implicitly by (27) below) this conformation tensor has the noteworthy property of being invariant with respect to conformal modifications of the background metric :

$$g_{\mu\nu} \mapsto \mathrm{e}^{2\alpha} g_{\mu\nu} \qquad \Rightarrow \qquad K_{\mu\nu}{}^{\rho} \mapsto K_{\mu\nu}{}^{\rho} + \eta_{\mu\nu} \perp^{\rho\sigma} \nabla_{\sigma}\alpha \,,$$

$$\mathcal{C}_{\mu\nu}{}^{\rho} \mapsto \mathcal{C}_{\mu\nu}{}^{\rho} \,. \tag{23}$$

This formula is useful [9] for calculations of the kind undertaken by Vilenkin [10] in a standard Robertson-Walker type cosmological background, which can be obtained from a flat auxiliary spacetime metric by a conformal transformation for which e^{α} is a time dependent Hubble expansion factor.

1.5 Codazzi, Gauss, and Schouten identities

As the higher order analogue of (14) we can go on to introduce the *third* fundamental tensor[4] as

$$\Xi_{\lambda\mu\nu}{}^{\rho} \overset{\text{def}}{=} \eta^{\sigma}{}_{\mu}\eta^{\tau}{}_{\nu}\perp^{\rho}_{\alpha}\overline{\nabla}_{\lambda}K_{\sigma\tau}{}^{\alpha}\,, \tag{24}$$

which by construction is obviously symmetric between the second and third indices and tangential on all the first three indices. In a spacetime background that is flat (or of constant curvature as is the case for the DeSitter universe model) this third fundamental tensor is fully symmetric over all the first three indices by what is interpretable as the *generalised Codazzi identity*.

In a background with arbitrary Riemann curvature $\mathcal{R}_{\lambda\mu}{}^{\rho}{}_{\sigma}$ the *generalised Codazzi identity* is expressible [3] as

$$\Xi_{\lambda\mu\nu}{}^{\rho} = \Xi_{(\lambda\mu\nu)}{}^{\rho} + \frac{2}{3}\eta^{\sigma}{}_{\lambda}\eta^{\tau}{}_{(\mu}\eta^{\alpha}{}_{\nu)}\mathcal{R}_{\sigma\tau}{}^{\beta}{}_{\alpha}\perp^{\rho}_{\beta} \qquad (25)$$

A script symbol $\mathcal{R}$ is used here in order to distinguish the (n-dimensional) background Riemann curvature tensor from the intrinsic curvature tensor (12) of the (d- dimensional) worldsheet to which the ordinary symbol R has already allocated. For many of the applications that will follow it will be sufficient just to treat the background spacetime as flat, i.e. to take $\mathcal{R}_{\sigma\tau}{}^{\beta}{}_{\alpha} = 0$.

For $\mathrm{n} > 2$, the background curvature tensor will be decomposible (if present) in terms of the background Ricci tensor and its scalar trace,

$$\mathcal{R}_{\mu\nu} = \mathcal{R}_{\rho\mu}{}^{\rho}{}_{\nu}\,, \qquad \mathcal{R} = \mathcal{R}^{\nu}{}_{\nu}\,, \tag{26}$$

and of its trace free conformally invariant Weyl part $\mathcal{W}_{\mu\nu}{}^{\rho}{}_{\sigma}$ – which can be non zero only for $\mathrm{n} \geq 4$ – in the well known [8] form

$$\mathcal{R}_{\mu\nu}{}^{\rho\sigma} = \mathcal{W}_{\mu\nu}{}^{\rho\sigma} + \frac{4}{\mathrm{n}-2} g^{[\rho}{}_{[\mu}\mathcal{R}^{\sigma]}{}_{\nu]} - \frac{2}{(\mathrm{n}-1)(\mathrm{n}-2)}\mathcal{R} g^{[\rho}{}_{[\mu} g^{\sigma]}{}_{\nu]}\,. \tag{27}$$

In terms of the tangential projection of this background curvature, the corresponding *internal* curvature tensor (12) takes the form

$$R_{\mu\nu}{}^{\rho}{}_{\sigma} = 2K^{\rho}{}_{[\mu}{}^{\tau}K_{\nu]\sigma\tau} + \eta^{\kappa}{}_{\mu}\eta^{\lambda}{}_{\nu}\mathcal{R}_{\kappa\lambda}{}^{\alpha}{}_{\tau}\eta^{\rho}{}_{\alpha}\eta^{\tau}{}_{\sigma}\,, \tag{28}$$

which is the translation into the present scheme of what is well known in other schemes as the *generalised Gauss identity*.

The less well known analogue (attributable [8] to Schouten) for the (trace free conformally invariant) $outer$ curvature (13) is expressible [3] in terms of the relevant projection of the background Weyl tensor as

$$\Omega_{\mu\nu}{}^{\rho}{}_{\sigma} = 2\, C_{[\mu}{}^{\tau\rho}\, C_{\nu]\tau\sigma} + \eta^{\kappa}{}_{\mu}\eta^{\lambda}{}_{\nu}\mathcal{W}_{\kappa\lambda}{}^{\alpha}{}_{\tau}\perp^{\rho}_{\alpha}\perp^{\tau}_{\sigma}\,. \qquad (29)$$

In a background that is flat or conformally flat (for which it is necessary, and for $\mathrm{n} \geq 4$ sufficient, that the Weyl tensor should vanish) the vanishing of the extrinsic conformation tensor $C_{\mu\nu}{}^{\rho}$ will therefore be sufficient (independently of the behaviour of the extrinsic curvature vector K^{μ}) for vanishing of the outer curvature tensor $\Omega_{\mu\nu}{}^{\rho}{}_{\sigma}$, which is the condition for it to be possible to construct fields of vectors λ^{μ} orthogonal to the surface and such as to satisfy the generalised Fermi-Walker propagation condition to the effect that $\perp^{\rho}_{\mu}\overline{\nabla}_{\nu}\lambda_{\rho}$ should vanish.

1.6 Internal Ricci and Conformal Curvatures.

The conclusion of the preceding paragraph is an illustration of the critically significant role of the conformation tensor $C_{\mu\nu}{}^{\rho}$ of an imbedding when the background is conformally flat, which suggests the interest of examining its role with respect to the *inner* curvature, $R_{\kappa\lambda}{}^{\mu}{}_{\nu}$ in this conformally flat case, for which the condition that the background Weyl tensor should vanish is necessary – and for $\mathrm{n} \geq 4$ also sufficient [8] – while when the background dimension is $\mathrm{n} = 3$ this condition, namely $\mathcal{W}_{\kappa\lambda}{}^{\mu}{}_{\nu} = 0\,,$ will will hold in any case as an identity. Conformal flatness is of course compatible with the most common applications, in which the background is taken to be flat in the strong sense, as a justifiable approximation in typical circumstances for which the characteristic length scales of the imbedding will be small compared with those of the background curvature.

Leaving aside the trivial (always locally conformally flat) case of a 2-dimensional background, the Gauss relation (28) reduces to

$$R_{\kappa\lambda}{}^{\mu}{}_{\nu} = \eta_{\kappa}{}^{\rho}\eta_{\lambda}{}^{\sigma}\mathcal{W}_{\rho\sigma}{}^{\tau}{}_{\upsilon}\eta^{\mu}{}_{\tau}\eta^{\upsilon}{}_{\nu} + 2K_{[\kappa}{}^{\mu\sigma}K_{\lambda]\nu\sigma}$$
$$+\frac{2}{\mathrm{n}-2}\Big(\eta_{[\kappa}{}^{\mu}\eta_{\lambda]}{}^{\rho}\eta_{\nu}{}^{\sigma} - \eta_{\nu[\kappa}\eta_{\lambda]}{}^{\rho}\eta^{\mu\sigma}\Big)\Big(\mathcal{R}_{\rho\sigma} - \frac{\mathcal{R}}{2(\mathrm{n}-1)}g_{\rho\sigma}\Big)\,.$$

The internal Ricci tensor is thus obtained in terms of the tracefree and trace parts $C_{\lambda\mu}{}^{\nu}$ and K_{ρ} of the second fundamental tensor in the form

$$R_{\mu\nu} = \mathcal{W}_{\mu\nu} - C_{\mu}{}^{\rho\sigma}C_{\nu\rho\sigma} + \frac{\mathrm{d}-2}{\mathrm{d}}C_{\mu\nu}{}^{\sigma}K_{\sigma} + \frac{\mathrm{d}-1}{\mathrm{d}^2}K^{\sigma}K_{\sigma}\,\eta_{\mu\nu}$$
$$+\frac{\mathrm{d}-2}{\mathrm{n}-2}\eta_{\mu}{}^{\rho}\eta_{\nu}{}^{\sigma}\mathcal{R}_{\rho\sigma} + \frac{1}{\mathrm{n}-2}\Big(\eta^{\rho\sigma}\mathcal{R}_{\rho\sigma} - \frac{\mathrm{d}-1}{\mathrm{n}-1}\mathcal{R}\Big)\,\eta_{\mu\nu}\,,$$

where the background Weyl contribution, if any, is given by the expressions

$$\mathcal{W}_{\mu\nu} = \eta_{\mu}{}^{\sigma}\eta_{\nu}{}^{\kappa}\,\mathcal{W}_{\rho\sigma}{}^{\tau}{}_{\kappa}\eta^{\rho}{}_{\tau} = -\eta_{\mu}{}^{\sigma}\eta_{\nu}{}^{\kappa}\,\mathcal{W}_{\rho\sigma}{}^{\tau}{}_{\kappa}\perp^{\rho}{}_{\tau}\,,$$

of which the last version is obtained as a consequence of the tracelessness of the Weyl tensor.

The corresponding Ricci scalar for the internal geometry (whose surface integral in the special (string) case $\mathrm{d}{=}2$ gives the ordinary Gauss Bonnet type invariant that was mentionned at the end of section 8) is thus finally obtained in the form

$$R = \mathcal{W} - \mathcal{C}_{\lambda\mu}{}^{\nu}\,\mathcal{C}^{\lambda\mu}{}_{\nu} + \frac{\mathrm{d}-1}{\mathrm{d}} K^{\sigma} K_{\sigma} + \frac{\mathrm{d}-1}{\mathrm{n}-2}\left(2\eta^{\rho\sigma}\mathcal{R}_{\rho\sigma} - \frac{\mathrm{d}}{\mathrm{n}-1}\mathcal{R}\right) ,$$

(which corrects a transcription error whereby a factor of two was omitted in the original version [3]) where the final Weyl contribution is just the trace,

$$\mathcal{W} = \mathcal{W}^{\nu}{}_{\nu} = \eta^{\rho\tau}\mathcal{W}_{\rho\sigma}{}^{\tau}{}_{\nu}\eta^{\nu}{}_{\tau} = \perp^{\rho\tau}\mathcal{W}_{\rho\sigma}{}^{\tau}{}_{\nu}\perp^{\nu}{}_{\tau} \,,$$

which can be seen to vanish identically unless both the dimension and the codimension of the worldsheet are greater than one, i.e. unless both $\mathrm{d}\geq 2$ and $\mathrm{n\text{-}d}\geq 2$.

For cases in which the imbedded surface has dimension $\mathrm{d} \leq 3$, as must always be the case in an ordinary 4-dimensional space-time background, the specification of the Ricci contribution provides all that is needed to specify the complete inner curvature tensor. However to fully specify $R_{\kappa\lambda}{}^{\mu}{}_{\nu}$ in higher dimensional cases for which the imbedded surface has dimension $\mathrm{d} \geq 4$ it will also be necessary to take account of the generically non zero conformal curvature term $\mathcal{C}_{\kappa\lambda}{}^{\mu}{}_{\nu}$ that will contribute to the total as given by the internal analogue of (27), namely

$$R_{\mu\nu}{}^{\rho\sigma} = \mathcal{C}_{\mu\nu}{}^{\rho\sigma} + \frac{4}{\mathrm{d}-2}\eta^{[\rho}{}_{[\mu}R^{\sigma]}{}_{\nu]} - \frac{2}{(\mathrm{d}-1)(\mathrm{d}-2)}R\eta^{[\rho}{}_{[\mu}\eta^{\sigma]}{}_{\nu]}\,, \qquad (30)$$

The rather greater algebraic effort required to work out this inner conformal curvature contribution is rewarded by the qualitatively tidy form of the result, which turns out to be homogeneously quadratic in the conformation tensor alone.

The contributions of the trace vector K^μ and of the background Ricci tensor $\mathcal{R}_{\mu\nu}$ are again (as in (29)) found to miraculously cancel out altogether, leaving

$$\mathcal{C}_{\kappa\lambda}{}^{\mu\nu} = 2\, C_{[\kappa}{}^{\mu\sigma}\, C_{\lambda]}{}^{\nu}{}_{\sigma} - \frac{4}{\mathrm{d}-2}\Big(C^{\rho[\mu}{}_{\sigma}\eta^{\nu]}{}_{[\kappa}\, C_{\lambda]\rho}{}^{\sigma} + \eta_{[\kappa}{}^{[\mu}\mathcal{W}_{\lambda]}{}^{\nu]}\Big)$$
$$- \frac{2}{(\mathrm{d}-2)(\mathrm{d}-1)}\eta_{[\kappa}{}^{\mu}\eta_{\lambda]}{}^{\nu}\Big(C_{\rho\sigma}{}^{\tau}\, C^{\rho\sigma}{}_{\tau} - \mathcal{W}\Big) + \eta_\kappa{}^\rho\eta_\lambda{}^\sigma\mathcal{W}_{\rho\sigma}{}^{\tau}{}_{\upsilon}\eta^{\mu}{}_{\tau}\eta^{\upsilon\nu}\ .$$

We can thus draw the memorable conclusion that in a conformally flat background the vanishing of the conformation tensor $C^{\mu\nu}{}_\rho$ is a sufficient condition not only for (local) outer flatness but also for (local) internal conformal flatness, at least for an imbedded surface with dimension $\mathrm{d}\geq 4$. With a little more work[3] it can be shown that this conclusion also holds for $\mathrm{d}{=}3$, while it is trivial for the case of a string worldheet $\mathrm{d}{=}2$, which is always (locally) conformally flat.

1.7 Special case of string worldsheet

The following work will deal mainly with the case $\mathrm{d}{=}2$ of a string, for which an orthonormal tangent frame will consist just of a timelike vector, ${\iota_0}^\mu$, and a spacelike vector, ${\iota_1}^\mu$. Their exterior product vector is the frame independent antisymmetric unit surface element tensor

$$\mathcal{E}^{\mu\nu} = 2{\iota_0}^{[\mu}{\iota_1}^{\nu]} = 2\big(-|\eta|\big)^{-1/2}\, x^{[\mu}{}_{,0}\, x^{\nu]}{}_{,1}\,, \tag{31}$$

whose tangential gradient satisfies

$$\overline{\nabla}_\lambda \mathcal{E}^{\mu\nu} = -2K_{\lambda\rho}{}^{[\mu}\mathcal{E}^{\nu]\rho}\,. \tag{32}$$

This is the special $\mathrm{d}{=}2$ case of formula (B9) in which a sign adjustment factor $(-1)^{\mathrm{d}-1}$ was omitted in the original analysis[3] (of which another misprint was omission in the first term of formula (10.4) of the factor 2).

In this $\mathrm{d}{=}2$ case the inner rotation pseudo tensor (10) is determined just by a corresponding rotation covector ρ_μ according to the specification

$$\rho_\lambda{}^\mu{}_\nu = \frac{1}{2}\,\mathcal{E}^\mu{}_\nu \rho_\lambda\,, \qquad \rho_\lambda = \rho_\lambda{}^\mu{}_\nu \mathcal{E}^\nu{}_\mu\,. \tag{33}$$

This can be used to see from (12) that the Ricci scalar,

$$R = R^\nu{}_\nu \qquad R_{\mu\nu} = R_{\rho\mu}{}^\rho{}_\nu\,, \tag{34}$$

of the 2-dimensional worldsheet will have the well known property of being a pure surface divergence, albeit of a frame gauge dependent quantity :

$$R = \overline{\nabla}_\mu\big(\mathcal{E}^{\mu\nu}\rho_\nu\big)\,. \tag{35}$$

In the specially important case of a string in ordinary 4-dimensional spacetime, i.e. when we have not only $\mathrm{d}{=}2$ but also $\mathrm{n}=4$, the antisymmetric background measure tensor $\varepsilon^{\lambda\mu\nu\rho}$ can be used to determine a scalar (or more strictly, since its sign is orientation dependent, a pseudo scalar) magnitude Ω for the outer curvature tensor (13) (despite the fact that its traces are identically zero) according to the specification

$$\Omega = \frac{1}{2}\,\Omega_{\lambda\mu\nu\rho}\,\varepsilon^{\lambda\mu\nu\rho}\,. \qquad (36)$$

Under these circumstances one can also define a "twist" covector ϖ_μ, that is the outer analogue of ρ_μ, according to the specification

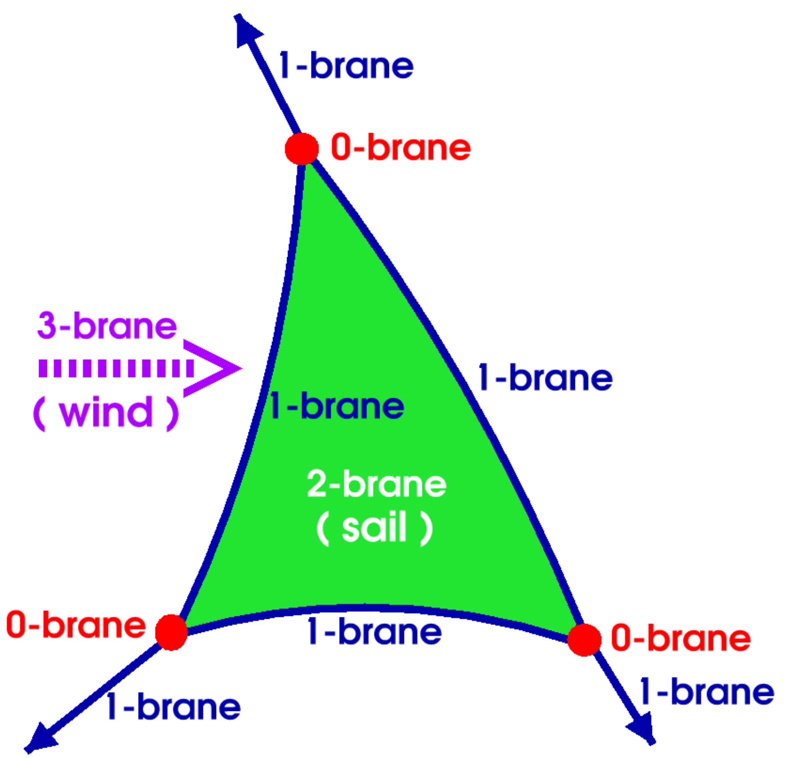
$$\varpi_\nu = \frac{1}{2}\,\varpi_\nu{}^{\mu\lambda}\,\varepsilon_{\lambda\mu\rho\sigma}\,\mathcal{E}^{\rho\sigma}\,. \qquad (37)$$

Using (37) it can be deduced from (13) that the outer curvature (pseudo) scalar Ω of a string worldsheet in 4-dimensions has a divergence property of the same kind as that of its more widely known Ricci analogue (35), the corresponding formula being given by

$$\Omega = \overline{\nabla}_\mu \big(\mathcal{E}^{\mu\nu} \varpi_\nu \big) \ . \tag{38}$$

It is to be remarked that for a compact spacelike 2-surface the integral of (32) gives the well known Gauss Bonnet invariant, but that the timelike string worldsheets under consideration here will not be characterised by any such global invariant since they will not be compact (being open in the time direction even for a loop that is closed in the spacial sense). The outer analogue of the Gauss Bonnet invariant that arises from (36) for a spacelike 2-surface has been discussed by Penrose and Rindler [11] but again there is no corresponding global invariant in the necessarily non-compact timelike case of a string worldsheet.

2 Laws of motion for a regular pure brane complex



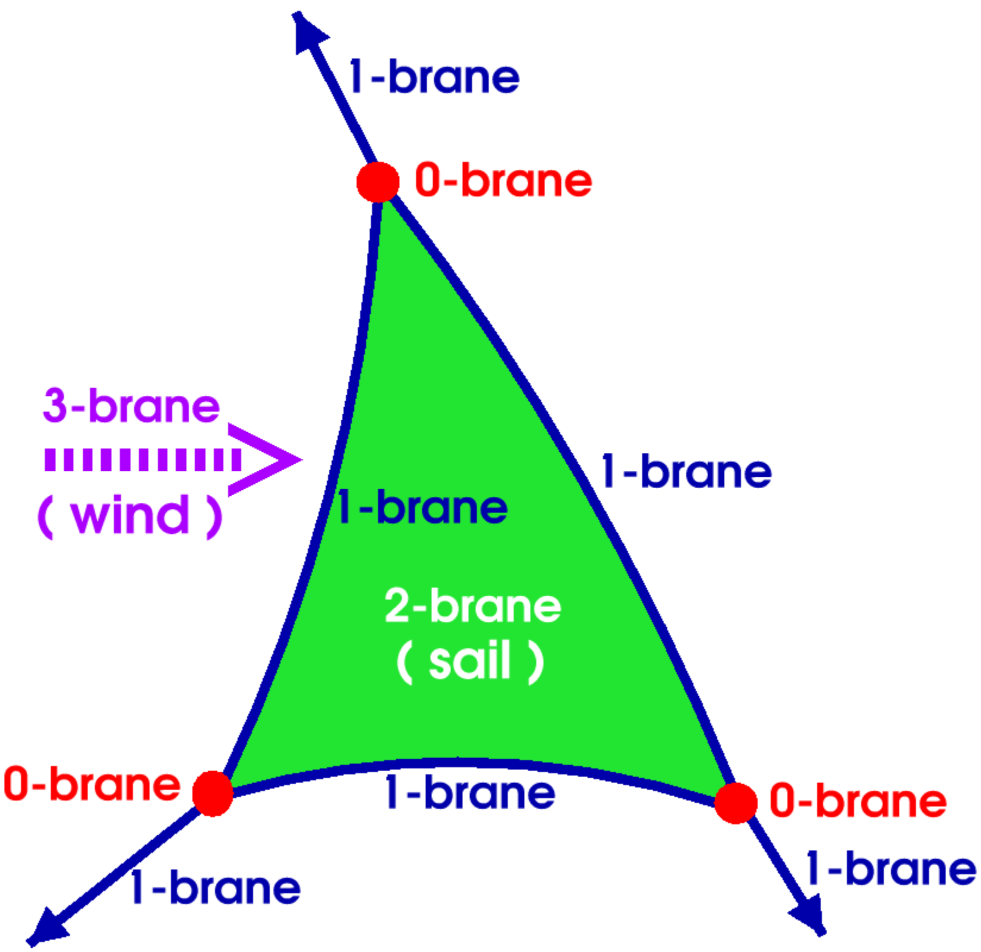
2.1 Definition of brane complex

The term p-brane has come [12], [13] to mean a dynamic system localised on a timelike support surface of dimension $d=p+1$, imbedded in a spacetime background of dimension $n > p$. Thus a zero - brane means what is commonly referred to as a "point particle", and a 1-brane means what is commonly referred to as a "string", while a 2-brane means what is commonly called a "membrane" (whence the generic term "brane").

At the upper extreme, the "improper" case of an (n-1)-brane is what is commonly referred to as a "medium" (as exemplified by a simple fluid). The codimension-1 (hypersurface supported) case of an (n-2)-brane (as exemplified by a cosmological domain wall) is what may be referred to as a "hypermembrane", while the codimension-2 case of an (n-3)-brane is what may analogously be referred to as a "hyperstring".

A set of branes of diverse dimensions will constitute a “geometrically regular” brane complex if the support surface of each $(\mathrm{d}\text{-}1)$-brane member is a smoothly imbedded d-dimensional timelike submanifold of which the boundary, if any, is a disjoint union of support surfaces of lower dimensional members of the set. For the complex to qualify as regular in the strong dynamic sense, it is also required that a p-brane member can act directly only on an $(\mathrm{p}\text{-}1)$-brane member on its boundary or on a $(p+1)$-brane member on whose boundary it is itself located, though it may be passively influenced by higher dimensional background fields.

Direct mutual interaction between branes with dimension differing by 2 or more will usually lead to divergences, symptomising the breakdown of a strict – meaning thin limit – brane description. To cure that properly, a more elaborate treatment – allowing for finite thickness – would be needed, but it may suffice to use a thin limit approximation wherby the divergence is absorbed in a renormalisation.



Nautical archetype of a regular brane complex in which a 3-brane (the wind) acts (by pressure discontinuity) on a 2-brane (the sail) hemmed by three 1-branes (bolt ropes) terminating on 0-branes (shackles) that are held in place by three more (free) 1-branes (external stay/sheet ropes).

In the case of a brane complex, the total action $\mathcal{I}$ will be given as a sum of contributions from the various $(\text{d-}1)$-branes of the complex, of which each has its own Lagrangian d-surface density scalar ${}^{(\text{d})}\overline{L}$ say. Each supporting d-surface will be specified by a mapping $\sigma \mapsto x\{\sigma\}$ giving the local background coordinates x^μ $(\mu = 0, ... , \text{n-}1)$ as functions of local internal coordinates σ^i $(i = 0, ... , \text{d-}1)$. The corresponding d-dimensional surface metric tensor ${}^{(\text{d})}\eta_{ij}$ induced as the pull back of the n-dimensional background spacetime metric $g_{\mu\nu}$, determines the surface measure, ${}^{(\text{d})}\text{d}\overline{S}$, in terms of which the total action will be expressible as

$$\mathcal{I} = \sum_{\text{d}} \int {}^{(\text{d})}\text{d}\overline{S}\ {}^{(\text{d})}\overline{L} \,, \qquad {}^{(\text{d})}\text{d}\overline{S} = \sqrt{\|{}^{(\text{d})}\eta\|}\, \text{d}^{\text{d}}\sigma \,. \tag{39}$$

For purposes such as the calculation of radiation, it may be useful to replace the *confined* (d-surface supported) but locally *regular* Lagrangian scalar fields ${}^{(\mathrm{d})}\overline{L}$ by corresponding unconfined, so no longer regular but *distributional* fields ${}^{(\mathrm{d})}\hat{L}$, so as to allow the total action (39) to be represented as a single background spacetime integral,

$$\mathcal{I} = \int \mathrm{d}\mathcal{S} \sum_{\mathrm{d}} {}^{(\mathrm{d})}\hat{L} \,, \qquad \mathrm{d}\mathcal{S} = \sqrt{\Vert g \Vert}\, \mathrm{d}^n x \,. \tag{40}$$

This requires the distributional action ${}^{(\mathrm{d})}\hat{L}$ for each $(\mathrm{d}\text{-}1)$-brane of the complex to be given in terms of the regular d-surface density scalar ${}^{(\mathrm{d})}\overline{L}$ by the prescription expressible in Dirac notation as

$${}^{(\mathrm{d})}\hat{L} = \Vert g \Vert^{-1/2} \int {}^{(\mathrm{d})}\mathrm{d}\overline{\mathcal{S}}\ {}^{(\mathrm{d})}\overline{L}\, \delta^n[x - x\{\sigma\}] \,. \tag{41}$$

2.2 Current, vorticity, and stress-energy tensor

As well as on its own internal (d-1)-brane surface fields and their derivatives, and those of any attached d-brane, each contribution ${}^{(\rm d)}\overline{L}$ will also depend (passively) on the spacetime metric $g_{\mu\nu}$ and perhaps other background fields, such as a Maxwellian gauge potential A_μ , or a generalised Kalb-Ramond gauge field $B^{[r]}_{\nu_1\ldots\nu_r} = B^{[r]}_{[\nu_1\ldots\nu_r]}$. In the unpolarised (fine) brane limit considered here the action will not depend on the background field derivatives. These give corresponding fields

$$F_{\mu\nu} = 2\nabla_{[\mu}A_{\nu]} \,, \qquad N^{[r+1]}_{\mu\nu_1\ldots\nu_r} = (r+1)\nabla_{[\mu}B^{[r]}_{\nu_1\ldots\nu_r]},$$

which are invariant under gauge changes

$$A_\mu \mapsto A_\mu + \nabla_\mu \alpha, \quad B^{[r]}_{\nu_1\nu_2\ldots\nu_r} \mapsto B^{[r]}_{\nu_1\nu_2\ldots.\nu_r} + r!\nabla_{[\nu_1}\chi_{\nu_2\ldots\nu_r]},$$

and are automatically closed :

$$\nabla_{[\rho}F_{\mu\nu]} = 0 \,, \quad \nabla_{[\rho}N^{[r+1]}_{\mu\nu_1\ldots\nu_r]} = 0 \,.$$

Subject to the internal dynamic equations of motion given by the variational principle stipulating preservation of the action by variations of the independent field variables, the effect of arbitrary infinitesimal "Lagrangian" variations $\underset{\rm L}{\delta} A_\mu$ $\underset{\rm L}{\delta} B^{[r]}_{\mu\nu}$, $\underset{\rm L}{\delta} g_{\mu\nu}$ of the background fields will be to induce a corresponding variation of the simple form

$$\delta\mathcal{I} = \sum_{\rm d} \int {}^{(\rm d)}{\rm d}\overline{\mathcal{S}} \Big({}^{(\rm d)}\overline{j}^{\mu} \underset{\rm L}{\delta} A_\mu + \frac{1}{r!}\, {}^{(\rm d)}\overline{\mathcal{W}}_{[r]}^{\nu_1\ldots\nu_r} \underset{\rm L}{\delta} B^{[r]}_{\nu_1\ldots\nu_r} + \tfrac{1}{2}\, {}^{(\rm d)}\overline{T}^{\mu\nu} \underset{\rm L}{\delta} g_{\mu\nu} \Big) ,$$

from which, for each $(d-1)$-brane, one can read out the electromagnetic surface current vector ${}^{(\rm d)}\overline{j}^{\mu}$, the surface flux (generalised vorticity) r-vector ${}^{(\rm d)}\overline{\mathcal{W}}_{[r]}^{\nu_1\ldots\nu_r} = {}^{(\rm d)}\overline{\mathcal{W}}_{[r]}^{[\nu_1\ldots\nu_r]}$, and, since $\underset{\rm L}{\delta}\big({}^{(\rm d)}{\rm d}\overline{\mathcal{S}}\big) = \tfrac{1}{2}\, {}^{(\rm d)}\eta^{\mu\nu}\big(\underset{\rm L}{\delta} g_{\mu\nu}\big)\, {}^{(\rm d)}{\rm d}\overline{\mathcal{S}}$, the surface stress momentum energy tensor

$${}^{(\rm d)}\overline{T}^{\mu\nu} = {}^{(\rm d)}\overline{T}^{\nu\mu} = 2\frac{\partial\, {}^{(\rm d)}\overline{L}}{\partial g_{\mu\nu}} + {}^{(\rm d)}\overline{L}\, {}^{(\rm d)}\eta^{\mu\nu} \, . \qquad (42)$$

2.3 Conservation of current and vorticity

Arbitrary gauge changes $\underset{\rm L}{\delta} A_\nu = \nabla_\nu \alpha$, and $\overset{[r]}{B}_{\nu_1\nu_2\dots} = r!\nabla_{[\nu_1}\chi_{\nu_2\dots]}$, with $\underset{\rm L}{\delta} g_{\mu\nu} = 0$, can only leave the action invariant

$$\sum_{\rm d} \int {\rm d}^{({\rm d})}\overline{\mathcal{S}} \left({}^{({\rm d})}\overline{j}^{\nu}\nabla_\nu \alpha + {}^{({\rm d})}\underset{[r]}{\overline{w}}^{\nu_1\nu_2\dots}\nabla_{\nu_1}\chi_{\nu_2\dots}\right) = 0,$$

if the current ${}^{({\rm d})}\overline{j}^{\mu}$ and vorticity flux ${}^{({\rm d})}\underset{[r]}{\overline{w}}^{\nu_1\nu_2\dots}$. are purely d-surface tangential : their contractions with the rank (n-d) orthogonal projector ${}^{({\rm d})}\!\perp^{\mu}_{\ \nu} = g^{\mu}_{\ \nu} - {}^{({\rm d})}\eta^{\mu}_{\ \nu}$ must vanish, ${}^{({\rm d})}\!\perp^{\mu}_{\ \nu}\,{}^{({\rm d})}\overline{j}^{\nu} = 0$, ${}^{({\rm d})}\!\perp^{\mu}_{\ \nu}\,{}^{({\rm d})}\underset{[r]}{\overline{w}}^{\nu\nu_2\dots} = 0$.

Thus decomposing the full gradient operator ∇_μ into tangential and orthogonally projected parts ${}^{({\rm d})}\overline{\nabla}_\mu = {}^{({\rm d})}\eta^{\nu}_{\ \mu}\nabla_\nu$ and ${}^{({\rm d})}\!\perp^{\nu}_{\ \mu}\nabla_\nu$, one sees that the gauge invariance condition for the action takes the form

$$\sum_{\rm d}\int {}^{({\rm d})}{\rm d}\overline{\mathcal{S}}\left(\alpha\,{}^{({\rm d})}\overline{\nabla}_\nu\,{}^{({\rm d})}\overline{j}^{\nu} + \chi_{\nu_2\dots}\,{}^{({\rm d})}\overline{\nabla}_\nu\,{}^{({\rm d})}\underset{[r]}{\overline{w}}^{\nu\nu_2\dots}\right) =$$

$$\sum_{\rm d}\int {}^{({\rm d})}{\rm d}\overline{\mathcal{S}}\,{}^{({\rm d})}\overline{\nabla}_\nu\left({}^{({\rm d})}\overline{j}^{\nu}\alpha + {}^{({\rm d})}\underset{[r]}{\overline{w}}^{\nu\nu_2\dots}\chi_{\nu_2\dots}\right),$$

in which integrands on the right are pure d-surface divergences.

For any d-dimensional support surface ${}^{(\mathrm{d})}\overline{\mathcal{S}}$, Green's theorem gives

$$\int {}^{(\mathrm{d})}d\overline{\mathcal{S}}\,{}^{(\mathrm{d})}\overline{\nabla}_\nu\,{}^{(\mathrm{d})}\overline{j}^{\,\nu} = \oint {}^{(\mathrm{d}-1)}d\overline{\mathcal{S}}\,{}^{(\mathrm{d})}\lambda_\nu\,{}^{(\mathrm{d})}\overline{j}^{\,\nu}\,, \tag{43}$$

taking the integral on the right over the boundary $(\mathrm{d}\text{-}1)$-surface of $\partial\,{}^{(\mathrm{d})}\overline{\mathcal{S}}$ of ${}^{(\mathrm{d})}\overline{\mathcal{S}}$, where ${}^{(\mathrm{d})}\lambda_\nu$ is the (uniquely defined) outward directed unit tangent vector on the d-surface at its $(\mathrm{d}\text{-}1)$-dimensional boundary. The gauge invariance condition on the action can thereby be reduced to

$$\sum_{\mathrm{p}} \int {}^{(\mathrm{p})}d\overline{\mathcal{S}}\,\Big\{\alpha\Big({}^{(\mathrm{p})}\overline{\nabla}_\nu\,{}^{(\mathrm{p})}\overline{j}^{\,\nu} - \sum_{\mathrm{d}=\mathrm{p}+1}{}^{(\mathrm{d})}\lambda_\nu\,{}^{(\mathrm{d})}\overline{j}^{\,\nu}\Big)$$
$$+\chi_{\nu_2\ldots}\Big({}^{(\mathrm{p})}\overline{\nabla}_\nu\,{}^{(\mathrm{p})}\overline{w}_{[r]}^{\,\nu\nu_2\cdots} - \sum_{\mathrm{d}=\mathrm{p}+1}{}^{(\mathrm{d})}\lambda_\nu\,{}^{(\mathrm{d})}\overline{w}_{[r]}^{\,\nu\nu_2\cdots}\Big)\Big\} = 0$$

where, for any p-dimensionally supported $(\mathrm{p}\text{-}1)$-brane, the summation "over $\mathrm{d}=\mathrm{p}+1$" is to be understood as consisting of a contribution from each $(\mathrm{p}+1)$-dimensionally supported p-brane attached to it.

The Maxwell gauge invariance condition (independence of α) is thus seen to be equivalent to the electric current conservation condition

$$
{}^{(\mathrm{p})}\overline{\nabla}_{\mu}\,{}^{(\mathrm{p})}\overline{j}^{\mu} = \sum_{\mathrm{d=p+1}} {}^{(\mathrm{d})}\lambda_{\mu}\,{}^{(\mathrm{d})}\overline{j}^{\mu}\,, \tag{44}
$$

which means that the source of charge injection into any particular $(\mathrm{p}\text{-}1)$-brane is the sum of the currents flowing in from the p-branes to which it is attached. The generalised Kalb-Ramond gauge invariance condition (independence of $\chi_{\nu_2\ldots\nu_r}$) can similarly be seen to be equivalent to the analogous (generalised vorticity) flux conservation condition

$$
{}^{(\mathrm{p})}\overline{\nabla}_{\mu}\,{}^{(\mathrm{p})}\overline{w}_{[r]}^{\;\mu\nu_2\ldots\nu_r} = \sum_{\mathrm{d=p+1}} {}^{(\mathrm{d})}\lambda_{\mu}\,{}^{(\mathrm{d})}\overline{w}_{[r]}^{\;\mu\nu_2\ldots\nu_r}\,. \tag{45}
$$

2.4 Force and the stress balance equation

The condition of being "Lagrangian" means that $\underset{\mathrm{L}}{\delta}$ is comoving as needed to be meaningful for fields with support confined to a particular brane. However for background fields one can also define an "Eulerian" variation, $\underset{\mathrm{E}}{\delta}$, with respect to some appropriately fixed reference system, in which the infinitesimal displacement of the brane complex is specified by a vector field ξ^μ . The difference will be given by

$$\underset{\mathrm{L}}{\delta} - \underset{\mathrm{E}}{\delta} = \vec{\xi}\!\!\pounds\,, \tag{46}$$

where the $\vec{\xi}\!\!\pounds$ is the Lie differentiation operator, which will be given for the relevant background fields by $\vec{\xi}\!\!\pounds A_\mu = \xi^\rho \nabla_\rho A_\mu + A_\rho \nabla_\mu \xi^\rho$, $\vec{\xi}\!\!\pounds B^{[r]}_{\mu\nu_2...} = \xi^\rho \nabla_\rho B^{[r]}_{\mu\nu_2...} + r B^{[r]}_{\rho[\nu_2...} \nabla_{\mu]} \xi^\rho$ and $\vec{\xi}\!\!\pounds g_{\mu\nu} = 2\nabla_{(\mu} \xi_{\nu)}$.

In a fixed Eulerian background, the background fields will have Lagrangian variations given just by their Lie derivatives with respect to the displacement ξ^μ . Subject to the internal field equations, the action variation $\delta\mathcal{I}$ due to the displacement of the branes will therefor just be $\sum_{\rm d}\int^{\rm (d)}\!{\rm d}\overline{\mathcal{S}}\left({}^{\rm (d)}\overline{j}^{\nu}\ \vec{\xi}\!\!\pounds A_\nu + \frac{1}{r!}\,{}^{\rm (d)}\overline{w}^{\mu\nu\dots}_{[r]}\ \vec{\xi}\!\!\pounds B^{[r]}_{\mu\nu\dots} + \frac{1}{2}\,{}^{\rm (d)}\overline{T}^{\mu\nu}\ \vec{\xi}\!\!\pounds g_{\mu\nu}\right)$.

The postulate that this vanishes for any ξ^μ entails the further $\rm d$-surface tangentiality restriction ${}^{\rm (d)}\!\perp^{\mu}{}_{\nu}\,{}^{\rm (d)}\overline{T}^{\nu\rho}=0$ and the requirement

$$\sum_{\rm d}\int^{\rm (d)}\!{\rm d}\overline{\mathcal{S}}\Big\{\xi^\rho\Big(F_{\rho\nu}\,{}^{\rm (d)}\overline{j}^{\nu}+\frac{1}{r!}N^{[r+1]}_{\rho\mu\nu\dots}\,{}^{\rm (d)}\overline{w}^{\mu\nu\dots}_{[r]}$$
$$-\,{}^{\rm (d)}\overline{\nabla}_{\mu}\,{}^{\rm (d)}\overline{T}^{\mu}{}_{\rho}-A_\rho\,{}^{\rm (d)}\overline{\nabla}_{\mu}\,{}^{\rm (d)}\overline{j}^{\mu}-\frac{1}{(r-1)!}B^{[r]}_{\rho\nu\dots}\,{}^{\rm (d)}\overline{\nabla}_{\mu}\,{}^{\rm (d)}\overline{w}^{\mu\nu\dots}_{[r]}\Big)$$
$$+\,{}^{\rm (d)}\overline{\nabla}_{\mu}\Big(\xi^\rho\big(A_\rho\,{}^{\rm (d)}\overline{j}^{\mu}+\frac{1}{(r-1)!}B^{[r]}_{\rho\nu\dots}\,{}^{\rm (d)}\overline{w}^{\mu\nu\dots}_{[r]}+{}^{\rm (d)}\overline{T}^{\mu}{}_{\rho}\big)\Big)\Big\}=0\quad,\qquad(47)$$

in which the final contribution is a pure surface divergence that can be dealt with using Green's theorem as before.

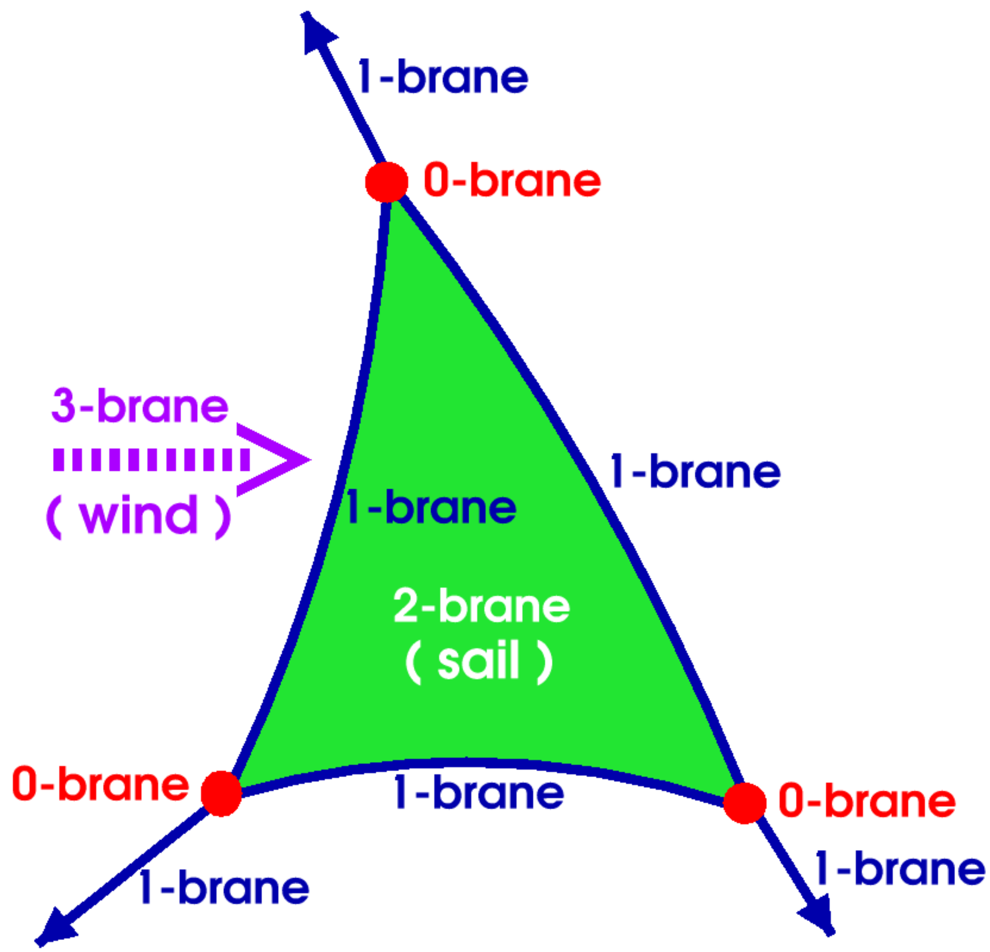
The conclusion is that invariance of the action under arbitrary displacements ξ^μ of the brane complex entails the dynamic equations

$$ {}^{(p)}\overline{\nabla}_\mu \, {}^{(p)}\overline{T}^\mu{}_\rho = {}^{(p)}f_\rho \, , \tag{48} $$

in which total force density, ${}^{(p)}f_\rho = {}^{(p)}\overline{f}_\rho + {}^{(p)}\check{f}_\rho$, includes Faraday-Lorenz and Joukowski-Magnus contributions from the background, as given by ${}^{(p)}\overline{f}_\rho = F_{\rho\mu} \, {}^{(p)}\overline{j}^\mu + \frac{1}{r!} N^{[r+1]}_{\rho\mu\nu\ldots} \, {}^{(p)}\overline{w}^{\mu\nu\ldots}_{[r]}$.
On each $(p\text{-}1)$-brane, the contact force exerted by attached p-branes is

$$ {}^{(p)}\check{f}_\rho = \sum_{d=p+1} {}^{(d)}\lambda_\mu \, {}^{(d)}\overline{T}^\mu{}_\rho \, , \tag{49} $$

in which it is to be recalled that, on the $(p{+}1)$-dimensional support surface of each attached p-brane, ${}^{(d)}\lambda_\mu$ is the unit vector that is directed normally towards the bounding $(p\text{-}1)$-brane.



Nautical archetype of a regular brane complex in which a 3-brane (the wind) acts (by pressure discontinuity) on a 2-brane (the sail) hemmed by three 1-branes (bolt ropes) terminating on 0-branes (shackles) that are held in place by three more (free) 1-branes (external stay/sheet ropes).

2.5 The equation of extrinsic motion

The tangential force balance equations will hold as identities when the internal field equations are satisfied (because a surface tangential displacement has no effect). The non-redundent information governing the extrinsic motion of a $(\mathrm{p}-1)$-brane will be given just by the orthogonal part. Integrating by parts, as the surface gradient of the rank-$(\mathrm{n}-\mathrm{p})$ orthogonal projector ${}^{(\mathrm{p})}\!\perp^{\mu}{}_{\nu}$ will be given in terms of the second fundamental tensor ${}^{(\mathrm{p})}\!K_{\mu\nu}{}^{\rho}$ of the p-surface by ${}^{(\mathrm{p})}\overline{\nabla}_{\mu}\,{}^{(\mathrm{p})}\!\perp^{\nu}{}_{\rho} = -\,{}^{(\mathrm{p})}\!K_{\mu\nu}{}^{\rho} - {}^{(\mathrm{p})}\!K_{\mu}{}^{\rho}{}_{\nu}\,,$ the extrinsic equations of motion are finally obtained in the form

$$ {}^{(\mathrm{p})}\overline{T}^{\mu\nu}\,{}^{(\mathrm{p})}\!K_{\mu\nu}{}^{\rho} = {}^{(\mathrm{p})}\!\perp^{\rho}{}_{\mu}\,{}^{(\mathrm{p})}\!f^{\mu}\,. \tag{50} $$

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3 Canonical Symplectic Structure

Abstract The covariant canonical variational procedure leading to the construction of a conserved bilinear symplectic current was originally developped in the context of field theory by Witten, Zuckerman, and others [1, 2, 3, 4, 5, 6, 7]. The following notes describe the generalisation of this procedure to brane mechanics in the manner initiated by Cartas-Fuentevilla [8, 9] and developed in collaboration with Dani Steer [10]. After a general presentation, including a review of the relationships between the various (Lagrangian, Eulerian and other) relevant kinds of variation, the procedure is illustrated by application to a particular category including the case of branes of purely elastic type.

3.1 Canonical formalism for Branes

Consider a generic conservative p-brane model whose mechanical evolution is governed by an action integral of the form

$$\mathcal{I} = \int \mathcal{L}\, \mathrm{d}^{\mathrm{p}+1}\sigma\,, \tag{51}$$

over a supporting worldsheet with internal co-ordinates σ^i $(i = 0, 1, \dots \mathrm{p})$, and induced metric $\eta_{ij} = g_{\mu\nu} x^\mu_{\,,i} x^\nu_{\,,j}$ in a background with coordinates x^μ, $(\mu = 0, 1, \dots \mathrm{n} - 1)$, $(\mathrm{n} \geq \mathrm{p} + 1)$ and (flat or curved) space-time metric $g_{\mu\nu}$.
The relevant Lagrangian scalar density is given by $\mathcal{L} = \Vert\eta\Vert^{1/2} L$, where L is scalar function of a set of field components q^{A} – including background coords – and of their surface deriatives, $q^{\mathrm{A}}_{\,,i} = \partial_i q^{\mathrm{A}} = \partial q^{\mathrm{A}} / \partial \sigma^i$.

The relevant field variables $q^{\rm A}$ can be of internal or external kind, the most obvious example of the latter kind being the background coordinates x^μ themselves.

The generic action variation, $\delta\mathcal{L} = \mathcal{L}_{\rm A}\delta q^{\rm A} + p_{\rm A}^{\ i}\delta q^{\rm A}_{\ ,i}$, specifies partial derivative components $\mathcal{L}_{\rm A}$ and and corresponding generalised momentum components $p_{\rm A}^{\ i}$. The variation principle characterises dynamically admissible "on shell" configurations by the vanishing of the Eulerian derivative

$$\frac{\delta\mathcal{L}}{\delta q^{\rm A}} = \mathcal{L}_{\rm A} - p_{\rm A}^{\ i}{}_{,i} \,. \qquad (52)$$

In terms of this Eulerian derivative, the generic Lagrangian variation will have the form

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta q^{\mathrm{A}}}\delta q^{\mathrm{A}} + \left(p_{\mathrm{A}}^{\ i}\delta q^{\mathrm{A}}\right)_{,i} . \tag{53}$$

There will be a corresponding pseudo-Hamiltonian scalar density

$$\mathcal{H} = p_{\mathrm{A}}^{\ i} q^{\mathrm{A}}_{\ ,i} - \mathcal{L} , \tag{54}$$

for which

$$\delta\mathcal{H} = q^{\mathrm{A}}_{\ ,i}\delta p_{\mathrm{A}}^{\ i} - \mathcal{L}_{\mathrm{A}}\delta q^{\mathrm{A}} . \tag{55}$$

(The covariance of such a pseudo-Hamiltonian distingushes it from the ordinary kind of Hamiltonian, which depends on the introduction of some preferred time foliation.)

For an on-shell configuration, i.e. when the dynamical equations

$$\frac{\delta \mathcal{L}}{\delta q^{\mathrm{A}}} = 0 \,, \tag{56}$$

are satisfied, the Lagrangian variation will reduce to a pure surface divergence,

$$\delta \mathcal{L} = \left(p_{\mathrm{A}}^{\ i} \delta q^{\mathrm{A}} \right)_{,i} \,, \tag{57}$$

and the correponding on-shell pseudo-Hamiltonian variation will take the form

$$\delta \mathcal{H} = q^{\mathrm{A}}_{\ ,i} \delta p_{\mathrm{A}}^{\ i} - p_{\mathrm{A}\ ,i}^{\ i} \delta q^{\mathrm{A}} \,. \tag{58}$$

3.2 Symplectic structure

The generic first order variation of the Lagrangian will be given by

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta q^{\mathrm{A}}}\delta q^{\mathrm{A}} + \vartheta^{i}{}_{,i}\,. \tag{59}$$

in terms of the generalised Liouville 1-form (on the configuration space cotangent bundle) defined by $\vartheta^{i} = p_{\mathrm{A}}^{\,i}\delta q^{\mathrm{A}}$.

Now consider a pair of successive independent variations $\acute{\delta}$, $\grave{\delta}$, which will give a second order variation of the form

$$\grave{\delta}\acute{\delta}\mathcal{L} = \grave{\delta}\Big(\frac{\delta\mathcal{L}}{\delta q^{\mathrm{A}}}\Big)\acute{\delta}q^{\mathrm{A}} + \frac{\delta\mathcal{L}}{\delta q^{\mathrm{A}}}\grave{\delta}\acute{\delta}q^{\mathrm{A}} + \big(\grave{\delta}p_{\mathrm{A}}^{\,i}\acute{\delta}q^{\mathrm{A}} + p_{\mathrm{A}}^{\,i}\grave{\delta}\acute{\delta}q^{\mathrm{A}}\big)_{,i}\,. \tag{60}$$

Thus using the commutation relation $\grave{\delta}\acute{\delta} = \acute{\delta}\grave{\delta}$ one gets

$$\grave{\delta}\Big(\frac{\delta\mathcal{L}}{\delta q^{\rm A}}\Big)\acute{\delta} q^{\rm A} - \acute{\delta}\Big(\frac{\delta\mathcal{L}}{\delta q^{\rm A}}\Big)\grave{\delta} q^{\rm A} = \hat{\varpi}^{i}{}_{,i}\,, \qquad (61)$$

where the symplectic 2-form (on the configuration space cotangent bundle) is defined by $\hat{\varpi}^{i} = \acute{\delta} p^{\,i}_{\rm A}\grave{\delta} q^{\rm A} - \grave{\delta} p^{\,i}_{\rm A}\acute{\delta} q^{\rm A}$.

For an on-shell perturbation we thus obtain

$$\frac{\delta\mathcal{L}}{\delta q^{\rm A}} = 0 \qquad \Rightarrow \qquad \delta\mathcal{L} = \vartheta^{i}{}_{,i}\,, \qquad (62)$$

while for a pair of on-shell perturbations we obtain

$$\acute{\delta}\Big(\frac{\delta\mathcal{L}}{\delta q^{\rm A}}\Big) = \grave{\delta}\Big(\frac{\delta\mathcal{L}}{\delta q^{\rm A}}\Big) = 0 \qquad \Rightarrow \qquad \hat{\varpi}^{i}{}_{,i} = 0\,. \qquad (63)$$

The foregoing surface current conservation law is expressible in shorthand as

$$\varpi^{i}{}_{,i} = 0\,, \tag{64}$$

in which the closed (since manifestly exact) symplectic 2-form (59) is specified in concise wedge product notation as

$$\varpi^{i} = \delta \wedge \vartheta^{i} = \delta p^{\,i}_{\mathrm{A}} \wedge \delta q^{\mathrm{A}}\,. \tag{65}$$

Some authors prefer to use an even more concise notation system in which it is not just the relevant distinguishing (in our case acute and grave accent) indices that are omitted but even the wedge symbol $\wedge$ that indicates the antisymmetrised product relation. However such an extreme level of abbreviation is dangerous [8] in contexts in which symmetric products are also involved.

3.3 Translation into strictly tensorial form.

To avoid the gauge dependence involved in the use of auxiliary structures such as local frames and internal surface coordinates, by working [11] just with quantities that are strictly tensorial with respect to the background space, one needs to replace the surface current densities whose components ϑ^i and ϖ^i depend on the choice of the internal coordinates σ^i, by vectorial quantities with strictly tensorial background coordinate components given by

$$\Theta^\nu = \|\eta\|^{-1/2} x^\nu{}_{,i} \vartheta^i \,, \qquad \Omega^\nu = \|\eta\|^{-1/2} x^\nu{}_{,i} \varpi^i \,.$$ and

with strictly divergences given by

$$\overline{\nabla}_\nu \Theta^\nu = \|\eta\|^{-1/2} \vartheta^i{}_{,i} \,, \qquad \overline{\nabla}_\nu \Omega^\nu = \|\eta\|^{-1/2} \varpi^i{}_{.i}$$

In terms of the surface projected covariant differentiation operator defined in terms of the fundamental tensor $\eta^{\mu\nu} = \eta^{ij} x^{\mu}{}_{,i} x^{\nu}{}_{,j}$ by $\overline{\nabla}_{\nu} = \eta^{\mu}{}_{\nu} \nabla_{\mu}$, one thus obtains a Liouville current conservation law of the form

$$\overline{\nabla}_{\nu} \Theta^{\nu} = 0 \qquad (66)$$

for any symmetry generating perturbation, i.e. for any infinitesimal variation δq^{A} such that $\delta \mathcal{L} = 0$.

Similarly a symplectic current conservation law of the form

$$\overline{\nabla}_{\nu} \Omega^{\nu} = 0 \qquad (67)$$

will hold for any pair of perturbations that are on-shell, i.e. such that $\delta(\delta \mathcal{L} / \delta q^{\mathrm{A}}) = 0$.

3.4 Covariant variation formulae

If the field q^{A} is defined over the background – not just confined to brane worldsheet with internal coordinates σ^i – then in terms of the relevant displacement vector, $\xi^\mu = \delta x^\mu$, with respect to a given (e.g. Minkowski type) system of background coordinates in terms of which $\partial_i q^{\mathrm{A}} = x^\mu_{\ ,i}\partial_\mu q^{\mathrm{A}}$, the simple worldsheet based field component variation δq^{A} will be given by

$$\delta q^{\mathrm{A}} = \underset{\mathrm{E}}{\delta} q^{\mathrm{A}} + \xi^\mu \partial_\mu q^{\mathrm{A}} . \tag{68}$$

where $\underset{\mathrm{E}}{\delta} q^{\mathrm{A}}$ is the relevant Eulerian variation, as defined with respect to the background system.

When one is dealing with a background field that is not simply a scalar but of a more general tensorial nature, it will commonly be desirable to go on to convert the Eulerian variation formula $\underset{\mathrm{E}}{\delta} = \delta - \vec{\xi}\cdot\partial$ into terms of covariant derivation as given by $\vec{\xi}\cdot\nabla = \vec{\xi}\cdot\partial + \{\vec{\xi}\cdot\Gamma\}$ where $\{\vec{\xi}\cdot\Gamma\}$ is a purely algebraic operator involving contractions with the 2-index quantity $(\vec{\xi}\cdot\Gamma)^{\mu}{}_{\nu} = \xi^{\rho}\Gamma_{\rho}{}^{\mu}{}_{\nu}\,,$ as exemplified, for a vectorial (e.g. Killing) field $k^{\mu}\,,$ or a covectorial (e.g. Maxwellian) form $A_{\mu}\,,$ by $\{\vec{\xi}\cdot\Gamma\}k^{\mu} = (\vec{\xi}\cdot\Gamma)^{\mu}{}_{\nu}k^{\nu}\,,$ and $\{\vec{\xi}\cdot\Gamma\}A_{\mu} = -(\vec{\xi}\cdot\Gamma)^{\nu}{}_{\mu}A_{\nu}\,.$

Instead of using the connection dependent covariant derivative, one can work with the corresponding Lie derivative, as given by a prescription of the form $\vec{\xi}\pounds = \vec{\xi}\cdot\nabla - \{\nabla\xi\}$, in which the operator $\{\nabla\xi\}$ acts by contractions with the displacement gradient tensor $\nabla_\nu\xi^\mu$, in the way exemplified respectively for a vector k^μ, or a 1-form (i.e. covector) A_μ, by the formulae $\{\nabla\xi\}k^\mu = k^\nu\nabla_\nu\xi^\mu$, and $\{\nabla\xi\}A_\mu = -A_\nu\nabla_\mu\xi^\nu$. It can be seen that connection cancels out, so that the Lie derivative will be equivalently expressible in terms just of partial derivative components $\partial_\nu\xi^\mu$ as

$$\vec{\xi}\pounds = \vec{\xi}\cdot\partial - \{\partial\xi\}\,. \tag{69}$$

Another kind of variation that is particularly important in the context of brane mechanics – because (unlike the Eulerian, covariant, and Lie derivatives) it is always well defined even for fields whose support is confined to the brane worldsheet – is what is known as the Lagrangian variation, meaning change with respect to background coordinates that are dragged by displacement. In the case of a field that is not confined to the brane worldsheet, so that its Eulerian variation is well defined, this latter kind will be related to the corresponding Lagrangian variation by the well known Lie derivation formula

$$\underset{\mathrm{L}}{\delta} = \underset{\mathrm{E}}{\delta} + \vec{\xi}\pounds \,. \tag{70}$$

Yet another possibility that may be useful is to express the Eulerian (fixed background point) variation in the form

$$\underset{\mathrm{E}}{\delta} = \underset{\Gamma}{\delta} - \vec{\xi} \cdot \nabla \, , \tag{71}$$

where parallely transported variation is defined – not just for background field, but also for tensor confined to brane – by

$$\underset{\Gamma}{\delta} = \delta + \{\vec{\xi} \cdot \Gamma\} \, , \tag{72}$$

using the operator notation introduced above. This parallel variation $\underset{\Gamma}{\delta}$ shares with the Lagrangian variation $\underset{\mathrm{L}}{\delta}$ the important property of being well defined not just for background fields but also for fields whose support is confined to the brane worldsheet.

The Lagrangian variation $\underset{\mathrm{L}}{\delta}$ will always be expressible directly in terms of the corresponding parallel variation $\underset{\Gamma}{\delta}$ by a relation of the form

$$\underset{\mathrm{L}}{\delta} = \underset{\Gamma}{\delta} - \{\nabla\vec{\xi}\}\,, \tag{73}$$

in which it can be seen that connection dependence cancels out, leaving an expression of the simple form

$$\underset{\mathrm{L}}{\delta} = \delta - \{\partial\vec{\xi}\}\,, \tag{74}$$

where the action of the algebraic operator $\{\partial\vec{\xi}\}$ is exemplified for a vector k^μ, or a covector A_μ, by the formulae $\{\partial\vec{\xi}\}k^\mu = k^\nu\partial_\nu\xi^\mu$, and $\{\partial\vec{\xi}\}A_\mu = -A_\nu\partial_\mu\xi^\nu$.

In conclusion of this overview of the relationships between the various kinds of infinitesimal variations that are commonly useful, it is to be mentionned that in literature dealing with purely non relativistic contexts in which it is possible (though not necessarily wise) to work exclusively with space coordinates of strictly Cartesian (orthonormal) type, the variations of the kind referred to here as “parallel” are generally described as “Lagrangian” by many authors. That usage does not necessarily lead to confusion, because for scalars the distinction does not arise, and because such authors systematically eschew the use (and the technical advantages) of Lagrangian variations of the fully comoving kind (that is considered here) by working exclusively with tensor components that are evaluated in terms only of orthonormal frames.

3.5 Evaluation via Lagrangian variations.

In typical applications, the relevant set of configuration components q^{A} will include a set of brane field components φ^{α} as well as the background coords x^{μ}, so that in terms of displacement vector $\xi^{\mu} = \delta x^{\mu}$ the Liouville current will take the form $\Theta^{\nu} = \|\eta\|^{-1/2} x^{\nu}{}_{,i}\big(p_{\alpha}{}^{i}\,\delta\varphi^{\alpha} + p_{\mu}{}^{i}\,\xi^{\mu}\big) = \pi_{\alpha}{}^{\nu}\,\delta\varphi^{\alpha} + \pi_{\mu}{}^{\nu}\,\xi^{\mu}$, in which the latter version replaces the original momentum components by the corresponding background tensorial momentum variables, which are given by $\pi_{\alpha}{}^{\nu} = \|\eta\|^{-1/2}\, x^{\nu}{}_{,i}\, p_{\alpha}{}^{i}$ and $\pi_{\mu}{}^{\nu} = \|\eta\|^{-1/2}\, x^{\nu}{}_{,i}\, p_{\mu}{}^{i}$.

To obtain an analogously tensorial formula for the symplectic current 2-form, it is convenient, as a first step, to take advantage of the symmetry property $\Gamma_{\mu}{}^{\nu}{}_{\rho} = \Gamma_{\rho}{}^{\nu}{}_{\mu}$, of the Riemannian background connection, which allows substitution of parallel variation $\underset{\Gamma}{\delta} p_{\mu}{}^{i} = \delta p_{\mu}{}^{i} - \Gamma_{\mu}{}^{\nu}{}_{\rho} p_{\nu}{}^{i} \xi^{\rho}$ for $\delta p_{\mu}{}^{i}$ so as to provide an expression of the form

$$\Omega^{\nu} = \|\eta\|^{-1/2} x^{\nu}{}_{,i} \left(\delta p_{\alpha}{}^{i} \wedge \delta \varphi^{\alpha} + \underset{\Gamma}{\delta} p_{\mu}{}^{i} \wedge \xi^{\mu} \right) . \tag{75}$$

The next step is to evaluate the relevant momentum variations in terms of the corresponding Lagrangian variations, using the formulae

$$\|\eta\|^{-1/2} x^{\nu}{}_{,i}\, \delta p_{\alpha}{}^{i} = \underset{\mathrm{L}}{\delta} \pi_{\alpha}{}^{\nu} + \pi_{\alpha}{}^{\nu} \overline{\nabla}_{\rho} \xi^{\rho} \,, \tag{76}$$

and

$$\|\eta\|^{-1/2} x^{\nu}{}_{,i}\, \underset{\Gamma}{\delta} p_{\mu}{}^{i} = \underset{\mathrm{L}}{\delta} \pi_{\mu}{}^{\nu} - \pi_{\rho}{}^{\nu} \nabla_{\mu} \xi^{\rho} + \pi_{\mu}{}^{\nu} \overline{\nabla}_{\rho} \xi^{\rho} \,. \tag{77}$$

The advantage of Lagrangian variations is their convenience for relating the relevant intrinsic physical quantities via the appropriate equations of state.

3.6 Application to hyperelastic case

The hyperelastic category [12] (generalising the case of an ordinary elastic solid which includes the special case of an ordinary barotropic perfect fluid) consists of brane models in which – with respect to a suitably comoving internal reference system σ^i – there are no independent surface fields at all – meaning that the φ^α and the ${p_\alpha}^i$ are absent – and in which the only relevant background field is the metric $g_{\mu\nu}$ that is specified as a function of the external coordinates x^μ . In any such case, the generic variation of the Lagrangian is determined just by the surface stress momentum energy density tensor $\overline{T}^{\mu\nu}$ according to the standard prescription $\delta\mathcal{L} = \frac{1}{2}\|\eta\|^{1/2}\,\overline{T}^{\mu\nu}\underset{\mathrm{L}}{\delta} g_{\mu\nu}\,,$ whereby $\overline{T}^{\mu\nu}$ is specified in terms of partial derivation of the action density with respect to the metric.

In a fixed background (i.e. in the absence of any Eulerian variation of the metric) the Lagrangian variation of the metric will be given, according to the formula (70), by $\underset{\rm L}{\delta} g_{\mu\nu} = \vec{\xi}\!\!\pounds g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$. Comparing this to canonical prescription $\delta\mathcal{L} = \mathcal{L}_\mu \xi^\mu + p_\mu{}^i \xi^\mu{}_{,i}$ with $\xi^\mu = \delta x^\mu$ shows that the relevant partial derivatives will be given by the (non-tensorial) formulae $\mathcal{L}_\mu = \Vert\eta\Vert^{1/2}\,\Gamma_\mu{}^\nu{}_\rho \overline{T}_\nu{}^\rho$ and $p_\mu{}^i = \Vert\eta\Vert^{1/2}\,\overline{T}_{\mu\nu}\eta^{ij} x^\nu{}_{,j}$.

It can thus be seen that in the hyperelastic case, the canonical momentum tensor $\pi_\mu{}^\nu$ and the Liouville current Θ^ν will be given just in terms of surface stress tensor $\overline{T}^{\mu\nu}$ by the very simple formulae

$$\pi_\mu{}^\nu = \overline{T}_\mu{}^\nu\,, \qquad \Theta^\nu = \overline{T}_\mu{}^\nu \xi^\mu\,. \tag{78}$$

In order to proceed, we must consider the second order metric variation, whereby (following Friedman and Schutz [13]) the hyper Cauchy tensor (generalised elasticity tensor) $\overline{\mathfrak{C}}^{\mu\nu\rho\sigma} = \overline{\mathfrak{C}}^{\rho\sigma\mu\nu}$ is specified [14] in terms of Lagrangian variations by a partial derivative relation of the form

$$\underset{\mathrm{L}}{\delta}\Big(\|\eta\|^{1/2}\,\overline{T}^{\mu\nu}\Big) = \|\eta\|^{1/2}\overline{\mathfrak{C}}^{\mu\nu\rho\sigma}\underset{\mathrm{L}}{\delta} g_{\rho\sigma}\,. \tag{79}$$

The symplectic current is thereby obtained in the form

$$\Omega^{\nu} = \mathfrak{D}_{\mu}{}^{\nu} \wedge \xi^{\mu}\,, \tag{80}$$

where

$$\mathfrak{D}_{\mu}{}^{\nu} = 2\overline{\mathfrak{C}}_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma}\overline{\nabla}_{\sigma}\xi^{\rho} + \overline{T}^{\nu\rho}\overline{\nabla}_{\rho}\xi_{\mu}\,. \tag{81}$$

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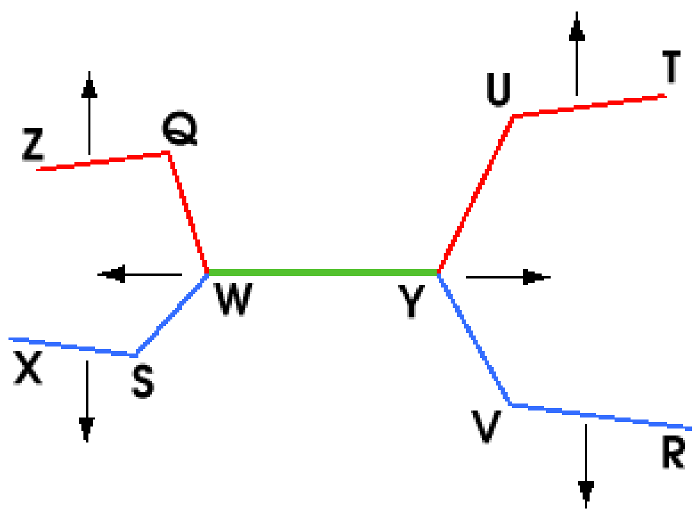
4 Application to string junctions and intercommutations



The prototype application is to a point particle, labelled by 0, at the junction between strings labelled by an index j that runs from 1 to 2 for a V junction, or from 1 to 3 for a Y junction (or even 1 to 4 for an X junction). The particle position x^μ will have proper time derivative $\dot{x}^\mu$ and acceleration $\ddot{x}^\mu = \dot{x}^\nu \nabla_\nu \dot{x}^\mu$ given, with $\dot{x}^\nu \ddot{x}_\nu = 0$, by

$$\eta_{\mathbf{0}}{}^{\sigma}{}_{\nu} \nabla_\sigma \overline{T}_{\mathbf{0}}{}^{\nu}{}_{\mu} = F_{\mu\nu} \overline{j}_{\mathbf{0}}^{\nu} + \sum_{\mathbf{j}} \lambda_{\mathbf{j}\nu} \overline{T}_{\mathbf{j}}{}^{\nu}{}_{\mu} \,, \tag{82}$$

in conjunction with the charge conservation law

$$\eta_{\mathbf{0}}{}^{\sigma}{}_{\nu} \nabla_\sigma \overline{j}_{\mathbf{0}}^{\nu} = \sum_{\mathbf{j}} \lambda_{\mathbf{j}\nu} \overline{j}_{\mathbf{j}}^{\nu} \,. \tag{83}$$

where the $\lambda_{\mathbf{j}\nu}$ are outward directed string tangent vectors, subject to orthonormality conditions $\lambda_{\mathbf{j}}^{\nu} \lambda_{\mathbf{j}\nu} = 1$, $\dot{x}^\nu \lambda_{\mathbf{j}\nu} = 0$, $\dot{x}^\nu \dot{x}_\nu = -1$.

The zero-brane fundamental tensor, energy tensor, and charge vector are given in terms of the particle mass $m_{\rm o}$ and charge number $z_{\rm o}$ by

$$\eta_{\rm o}^{\mu\nu} = -\dot x^\mu \dot x^\nu , \quad \overline{T}_{\rm o}^{\mu\nu} = m_{\rm o} \dot x^\mu \dot x^\nu , \quad \overline{j}_{\rm o}^{\nu} = e\, z_{\rm o} \dot x^\nu ,$$

where e is a charge coupling constant, while $K_{{\rm o}\mu\nu}{}^\rho = -\eta_{{\rm o}\mu\nu}\, \ddot x^\rho$.

We thus get $\dot m_{\rm o} \dot x_\mu + m_{\rm o} \ddot x_\mu - e\, z_{\rm o} F_{\mu\nu} \dot x^\nu = \sum_{\rm j} \lambda_{{\rm j}\nu} \overline{T}_{{\rm j}\,\mu}^{\nu}$,

and $e\, \dot z_{\rm o} = \sum_{\rm j} \lambda_{{\rm j}\nu} \overline{j}_{\rm j}^{\nu}$, while as a consequence we shall have $\dot m_{\rm o} = -\dot x^\mu \sum_{\rm j} \lambda_{{\rm j}\nu} \overline{T}_{{\rm j}\,\mu}^{\nu}$. If no strings attached, right hand sides will drop out, so the proper time derivatives $\dot m_{\rm o}$ and $\dot z_{\rm o}$ must vanish, leaving a dynamic equation of the familiar form $m_{\rm o} \ddot x_\mu = e\, z_{\rm o} F_{\mu\nu} \dot x^\nu$.

Our concern here is with the opposite extreme, for which the left hand sides drop out in the absence of a substantial particle at the junction, i.e. when $m_{\rm o} = 0$ and $z_{\rm o} = 0$, so that we shall be left with the junction conditions $\sum_{\rm j} \lambda_{{\rm j}\nu} \overline{T}_{{\rm j}\,\mu}^{\nu} = 0$ and $\sum_{\rm j} \lambda_{{\rm j}\nu} \overline{j}_{\rm j}^{\nu} = 0$.

For any string model the 2 dimensional worldsheet will have unit surface bivector and (first) fundamental tensor $\eta^\mu{}_\nu = \mathcal{E}^\mu{}_\rho \mathcal{E}^\rho{}_\nu$ expressible in terms of any orthonormal tangent diad u^ν, $\widetilde{u}^\nu$ as $\mathcal{E}^{\mu\nu} = 2\, u^{[\mu}\, \widetilde{u}^{\nu]}$ and $\eta^\mu{}_\nu = -\, u^\mu\, u^\nu + \widetilde{u}^\mu \widetilde{u}^\nu$. The symmetric surface stress energy tensor will be expressible in the form

$$\overline{T}^{\mu\nu} = \beta_+^{(\mu} \beta_-^{\nu)} . \tag{84}$$

in terms of a pair of bicharacteristic vectors having the generically timelike form $\beta_\pm^\mu = \sqrt{\mathcal{U}}\, u^\mu \pm \sqrt{\mathcal{T}}\, \widetilde{u}^\mu$, in terms of a preferred diad such that

$$\overline{T}^{\mu\nu} = \mathcal{U}\, u^\mu\, u^\nu - \mathcal{T}\, \widetilde{u}^\mu\, \widetilde{u}^\nu \tag{85}$$

where $\mathcal{U}$ is the surface energy density and $\mathcal{T}$ the string tension.

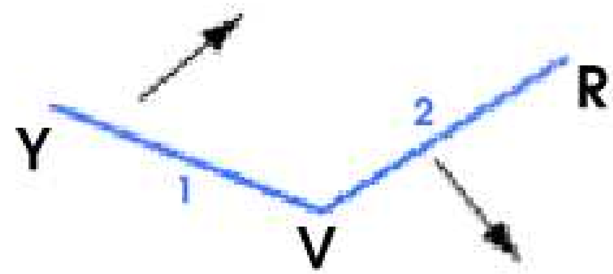
The magnitudes of the bicharacteristic vectors will then be given by $\beta_{+\mu}\beta_{+}{}^{\mu} = \beta_{-\nu}\beta_{-}{}^{\nu} = -(\mathcal{U} - \mathcal{T})$, while their scalar product would any case be given by $\beta_{+\mu}\beta_{-}^{\mu} = \overline{T}^{\nu}{}_{\nu} = -\big(\mathcal{U} + \mathcal{T}\big)$.

At the junction, for the j th string, we shall have $\boldsymbol{u}_{\mathbf{j}}^{\nu} = \gamma_{\mathbf{j}}(\dot{x}^{\nu} - v_{\mathbf{j}}\lambda_{\mathbf{j}}^{\nu})$ and $\widetilde{\boldsymbol{u}}_{\mathbf{j}}^{\nu} = \gamma_{\mathbf{j}}(\lambda_{\mathbf{j}}^{\nu} - v_{\mathbf{j}}\dot{x}^{\nu})$ where $\gamma_{\mathbf{j}} = (1 - v_{\mathbf{j}}^{2})^{-1/2}$, so with $\beta_{\mathbf{j}\pm}^{\nu} = \beta_{\mathbf{j}\pm}^{0}\dot{x}^{\nu} + \beta_{\mathbf{j}\pm}^{1}\lambda_{\mathbf{j}}^{\nu}$ its force contribution will be

$$\overline{T}_{\mathbf{j}}^{\mu\nu}\lambda_{\mathbf{j}\nu} = \gamma_{\mathbf{j}}^{2}(\mathcal{U}_{\mathbf{j}}v_{\mathbf{j}}^{2} - \mathcal{T}_{\mathbf{j}})\lambda_{\mathbf{j}}^{\nu} - \gamma_{\mathbf{j}}^{2}v_{\mathbf{j}}(\mathcal{U}_{\mathbf{j}} - \mathcal{T}_{\mathbf{j}})\dot{x}^{\nu} , \qquad (86)$$

$$= \beta_{\mathbf{j}+}^{1}\beta_{\mathbf{j}-}^{1}\lambda_{\mathbf{j}}^{\mu} + \beta_{\mathbf{j}+}^{(1}\beta_{\mathbf{j}-}^{0)}\dot{x}^{\mu} .$$

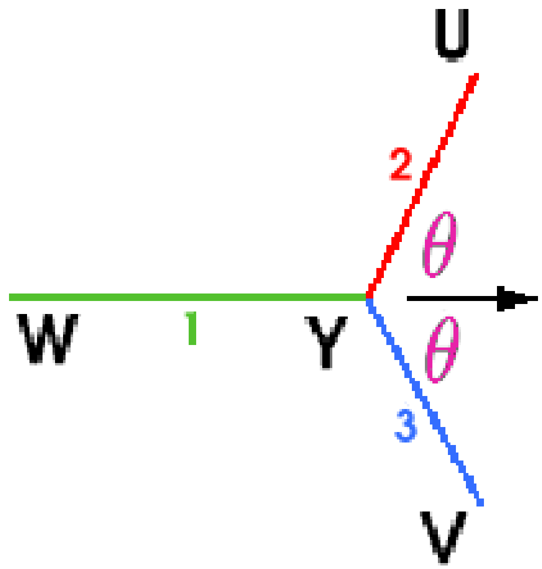
At a V junction between just 2 string segments in different directions, $\lambda_{\mathbf{1}}^{\mu} \neq -\lambda_{\mathbf{2}}^{\mu}$, the coefficient of the first term in (86) and thus one of its factors, must vanish : $\beta_{\mathbf{j}+}^{1} = 0$ say, so the junction worldline is itself bicharacteristic : $\dot{x}^{\nu} \propto \beta_{\mathbf{j}+}^{\nu}$.



On either side of a V junction (kink) between string segments YV and VR the relative flow speed v must satisfy the condition $v^2 = c_{\rm E}{}^2$ where $c_{\rm E} = \sqrt{\mathcal{T}/\mathcal{U}}$ is the extrinsic (wiggle) propagation speed. The force balance condition is then just that there be no jump discontinuity in the energy transmission rate, which means

$$\left[\sqrt{\mathcal{U}\mathcal{T}}\,\right]_1^2 = 0\,. \qquad (87)$$

If the string state depends on electric and/or other surface currents, their conservation conditions will just be equivalent to continuity of the relevant variables.



Now consider a Y junction (bifurcation) where a first string segment WY splits into two branches YV and YU , which (for simplicity) we suppose to be symmetric, both deviating by the same angle θ from the direction of WY in the junction rest frame, so that $\lambda^{\mu}_{2} + \lambda^{\nu}_{3} = -2\cos\theta\,\lambda^{\nu}_{1}$.
Dropping the suffices 2,3 for the symmetrically related branches, the force balance conditions will be expressible as

$$\gamma_{1}^{2} v_{1} (\mathcal{U}_{1} - \mathcal{T}_{1}) = -2\gamma^{2} v (\mathcal{U} - \mathcal{T})\,, \qquad (*)$$

and

$$\gamma_{1}^{2} (\mathcal{U}_{1} v_{1}^{2} - \mathcal{T}_{1}) = 2\cos\theta\,\gamma^{2}\,(\mathcal{U} v^{2} - \mathcal{T})\,. \quad (a)$$

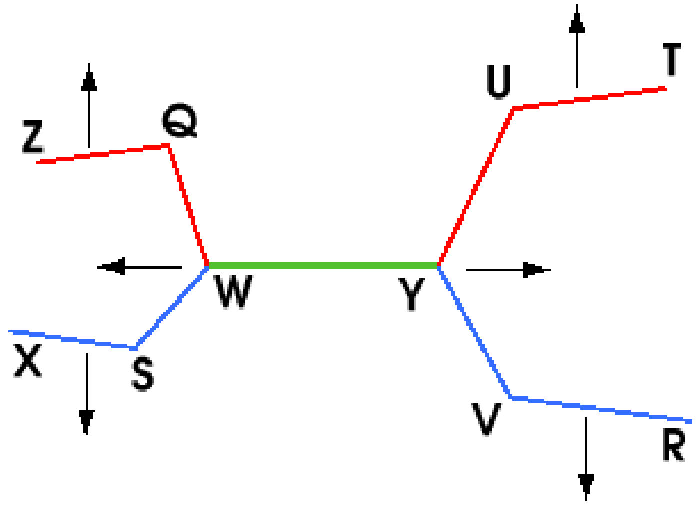
If there is a current $\nu^{\mu} = \nu\, \boldsymbol{u}^{\mu}$ of particles that are conserved not just internally but also at at junctions, then in terms of the specific enthalpy, $h = (\mathcal{U} - \mathcal{T})/\nu$, the energy conservation condition (*) will reduce to

$$\gamma_{_1} h_{_1} = \gamma\, h\,, \qquad (b)$$

by the condition $\sum_{\rm j} \lambda_{{\rm j}\nu} \nu_{\rm j}^{\nu} = 0$, which, for symmetric Y-junction, is

$$\gamma_{_1} v_{_1} \nu_{_1} = -2\,\gamma\, v\, \nu\,. \qquad (c)$$

In the ordinary elastic case, for which $\mathcal{U}$ depends just on $\mathcal{T}$, such a number density ν and the corresponding chemical potential or effective mass $\mu = {\rm d}\mathcal{U}/{\rm d}\nu$, are specifiable by $\ln\{\nu\} = \int {\rm d}\mathcal{U}/(\mathcal{U} - \mathcal{T})$, with the identification $\mu = h$. This gives two conserved currents, $\nu^{\mu} = \nu\, \boldsymbol{u}^{\mu}$ and $\widetilde{\mu}^{\nu} = \mu\, \widetilde{\boldsymbol{u}}^{\nu}$ (of which one or other will be identifiable, in the electromagnetic case enviseaged by Witten, with the current $\boldsymbol{z}^{\nu}$ above). The two equations (b) and (c) suffice to determine the change in velocity and state in the elastic case, and the preceding equation (a) then fixes the angle θ.



As in hep-th/0601153, using frame with unit vectors $\boldsymbol{e}_t^{\mu}$, $\boldsymbol{e}_x^{\mu}$, $\boldsymbol{e}_y^{\mu}$, $\boldsymbol{e}_z^{\mu}$, consider symmetric collision of strings ZT and XR with directions deviating from x direction by angles $\pm\alpha$ in y direction, and velocities $\pm\,\boldsymbol{v}_z$ in z direction, after formation of connecting segment WY at rest in the x direction. In terms of time t and internal space coordinate σ, the unperturbed segments ZQ and UT of the first string will thus have position given by

$$x^{\mu} = t\,(\,\boldsymbol{e}_t^{\mu} + \,\boldsymbol{v}_z\,\,\boldsymbol{e}_z^{\mu}) + \sigma(\cos\alpha\,\,\boldsymbol{e}_x^{\mu} + \sin\alpha\,\,\boldsymbol{e}_y^{\mu})\,,$$

with preferred internal frame given by $\boldsymbol{u}^{\mu} = \gamma_z(\,\boldsymbol{e}_t^{\mu} + \,\boldsymbol{v}_z\,\,\boldsymbol{e}_z^{\mu})\,.$

Relative to the preferred frame, the kink U has characteristic speed $c_{\rm E}$, and so is given by $\sigma = \gamma_z^{-1} c_{\rm E}\, t$. This gives the unit bicharacteristic :

$$\beta_+^{\nu} \propto \; u_+^{\nu} = \gamma_{\rm E}\gamma_z(\, e_t^{\mu} + v_z \; e_z^{\nu}) + c_{\rm E}\gamma_{\rm E}(\cos\alpha \; e_x^{\nu} + \sin\alpha \; e_y^{\nu}) \,.$$

The connecting string segment WY will have tangent vector $\lambda_1^{\nu} = \gamma_1(\, e_x^{\nu} + v_1 \; e_t^{\nu})$ normal to the worldline of the junction Y, of which the tangent $\dot{x}^{\nu} = \gamma_1(\, e_t^{\nu} + v_1 \; e_x^{\nu})$ is also, like u_+^{ν}, tangent to the segment YU, in which, with $\gamma_+ = -\dot{x}_{\nu}\, u_+^{\nu} = \gamma_{\rm E}\gamma_1(\gamma_z - c_{\rm E} v_1 \cos\alpha)$, the normal frame vector will be $\lambda^{\nu} = (\gamma_+^2 - 1)^{-1/2}(\gamma_+\, \dot{x}^{\nu} - \; u_+^{\nu})$. The prefered frame vector with speed $c_{\rm E}$ relative to u_+^{ν} in YU is thus

$$u^{\nu} = \gamma_{\rm E}(\gamma_+ - c_{\rm E}\sqrt{\gamma_+^2 - 1})(\dot{x}^{\nu} - v\,\lambda^{\nu})$$ with velocity

$$v = (\, v_+ - c_{\rm E})/(1 - c_{\rm E}\; v_+) \quad \text{where} \quad v_+ = \sqrt{1 - \gamma_+^{-2}}\,,$$

which is needed for (b) and (c). Finally the angle needed for (a) is given by

$$\cos\theta = -\lambda_1^{\nu}\lambda_{\nu} = \gamma_{\rm E}\gamma_1(\gamma_+^2 - 1)^{-1/2}(\, c_{\rm E}\cos\alpha - v_1\gamma_z) \,.$$

Eliminating θ, v, and its Lorenz factor $\gamma = \gamma_{\rm E}\gamma_+(1 - c_{\rm E}\, v_+)$,
from (*) and (a) thus gives the two conditions

$$\gamma_1^2 v_1(\mathcal{U}_1 - \mathcal{T}_1) = 2\gamma_+^2(c_{\rm E} - v_+)(1 - c_{\rm E}\, v_+)\mathcal{U}\,, \quad \text{and}$$

$$\gamma_1(\mathcal{U}_1 v_1^2 - \mathcal{T}_1) = 2\gamma_{\rm E}\gamma_+\Big((1 + c_{\rm E}^2)\, v_+ - 2\, c_{\rm E}\Big)(c_{\rm E}\cos\alpha - v_1\gamma_z)\mathcal{U},$$

which suffices for determination on WY of v_1 and $\mathcal{T}_1$ provided the latter is given as a function of $\mathcal{U}_1$ by some equation of state.

A prototypical example is provided by warm string model for thermal distribution of wiggles with temperature Θ and entropy density s on an underlying Nambu-Goto model with Kibble mass m $(= \sqrt{\hbar\text{“}\mu\text{”}})$ meaning that $\mathcal{T} = \mathcal{U} = m^2/\hbar$, for which macroscopic averaging gives

$$\frac{\hbar\mathcal{T}}{m^2} = \frac{m^2}{\hbar\mathcal{U}} = c_{\rm E} = \sqrt{1 - \frac{2\pi\Theta^2}{3\, m^2}} = \left(1 + \frac{3\hbar^2 s^2}{2\pi\, m^2}\right)^{-1/2}. \qquad (88)$$

In the ultrarelativistic Nambu-Goto limit, $\mathcal{U} - \mathcal{T} \to 0\,, \quad c_{\rm E} \to 1\,,$ $\gamma_{\rm E} = (1 - c_{\rm E}{}^2)^{-1/2} \to \infty\,,$ there is no current and remaining Y junction condition is (a), which reduces to $\mathcal{T}_1 = 2\cos\theta\,\mathcal{T}$ (as for ordinary static equilibrium) while preceding formula simply gives $\cos\theta = (\cos\alpha - v_1\gamma_z)/(\gamma_z - v_1\cos\alpha)\,.$ We thus recover the Copeland-Kibble-Steer formula for speed of Y junction along x axis :

$$v_1 = \frac{2\mathcal{T}\cos\alpha - \mathcal{T}_1\,\gamma_z}{2\mathcal{T}\,\gamma_z - \mathcal{T}_1\cos\alpha}\,. \qquad (89)$$

In generic elastic case, $\mathcal{T}_1$ is not fixed, but depends on internal state of connecting string, WY. This will be determined by 2 more equations, (b) and (c). That would be OK for static equilibrium with an adjustible angle, but it over-determines the case of a dynamic collision with α given in advance. So (as in an ordinary shock) treatment of such a collison will generically require use of a non-conservative model !

As 2-dim analogue of ordinary perfect fluid, non-conservative string models have energy density $\mathcal{U}$ depending, not just on conserved number density ν, but also on another number density s representing entropy, subject to $\gamma_1 v_1 s_1 + 2\,\gamma\, v\, s \geq 0$. Thus (generalising warm string limit with $\nu = 0$) the generic variation, $\mathrm{d}\mathcal{U} = \mu\,\mathrm{d}\nu + \Theta\,\mathrm{d}s$, specifies chemical potential μ (i.e. effective mass per particle) and a thermal potential Θ (i.e temperature) on the string, whose tension will be $\mathcal{T} = \mathcal{U} - \mu\nu - \Theta s$, while enthalpy per particle in (b) will then be $h = \mu + \Theta s/\nu$. The Y junction condition is given, for $\iota = 2$, by

$$(\mathcal{T}_1 - \iota\cos\theta\,\mathcal{T})\left(\frac{h_1}{\nu_1} + \frac{h}{\iota\nu}\right) = (h^2 - h_1{}^2)\,\frac{\iota\nu\, h_1 - \cos\theta\,\nu_1 h}{\iota\nu\, h_1 - \nu_1 h}. \quad (90)$$

(Taub shock condition in unbent string given by $\iota = 1$ with $\theta = 0$.)

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5 Dynamics and vorton equilibrium states of elastic string loops

1. Kinematics of thin string or brane

Classical p-brane model qualifies as 'thin' if support confined near timelike worldsheet of dim p+1, coords σ^i , i= 0,1, ... p, with p=1 in case of string.

In n dim background, with coords x^μ , $\mu = 0, 1, ...\mathrm{n} - 1$, metric $g_{\mu\nu}$, the brane embedding induces worldsheet metric $\eta_{ij} = g_{\mu\nu} x^\mu_{,i} x^\nu_{,j}$, whose contravariant inverse induces (first) fundamental tensor $\eta^{\mu\nu} = \eta^{ij} x^\mu_{,i} x^\nu_{,j}$ giving tangential projection tensor $\eta^\nu_\mu = g^\nu_\mu - \perp^\nu_\mu$ and tangential deriv operator $\overline{\nabla}_\mu = \eta^\nu_\mu \nabla_\nu$.

Hence construct second fundamental tensor $K_{\mu\nu}{}^\rho = \eta^\sigma_\nu \overline{\nabla}_\mu \eta^\rho_\sigma$, with symmetry $K_{[\mu\nu]}{}^\rho = 0$ as condition for integrability, and projection properties $\perp^\sigma_\mu K_{\sigma\nu}{}^\rho = 0 = K_{\mu\nu}{}^\sigma \eta^\rho_\sigma$.

2. Dynamics of thin string or brane

Brane governed by worldsheet action $\mathcal{I} = \int L \sqrt{-\eta}\, \mathrm{d}^{\mathrm{p}+1}\sigma$ will have surface stress energy tensor given by $\overline{T}^{\mu\nu} = 2\,\partial L/\partial g_{\mu\nu} + L\eta^{\mu\nu}$.
Evolution of worldsheet subject to external surface force density $\overline{f}^{\mu}$ given by universally applicable dynamical equation $\overline{T}^{\mu\nu} K_{\mu\nu}{}^{\rho} = \perp^{\rho}_{\sigma} \overline{f}^{\sigma}$.

In generic string case, $\exists$ orthonormal timelike and spacelike eigenvectors $u^{\mu} = x^{\mu}{}_{,i} u^{i}$, $\tilde{u}^{\mu} = x^{\mu}{}_{,i} \tilde{u}^{i}$, $\tilde{u}^{i} = \varepsilon^{ij} u_{j}$ associated with energy density $\mathcal{U}$ and string tension $\mathcal{T}$ such that $\eta^{\mu\nu} = -u^{\mu}u^{\nu} + \tilde{u}^{\mu}\tilde{u}^{\nu}$ and $\overline{T}^{\mu\nu} = \mathcal{U} u^{\mu}u^{\nu} - \mathcal{T} \tilde{u}^{\mu}\tilde{u}^{\nu}$.
In particular, a Nambu Goto (internally isotropic) string has $\mathcal{U} = \mathcal{T} = m^{2}$ for some fixed mass m , so in terms of worldsheet curvature vector $K^{\rho} = K^{\nu}{}_{\nu}{}^{\rho}$ its dynamical equation will just be $K^{\rho} = m^{-2} \perp^{\rho}_{\sigma} \overline{f}^{\sigma}$.

3. Free motion of elastic string models

Witten's conduction mechanism provides string Lagrangian L depending only on scalar $w = \gamma^{ij}\varphi_{,i}\varphi_{,j}$, providing adjoint $\Lambda = L + w\kappa$ with $\kappa = -2\,\mathrm{d}L/\mathrm{d}w$, in terms of which transverse (wiggle) and longit (sound) propagation speeds are respectively $v = \sqrt{\mathcal{T}/\mathcal{U}}$ and $v_{\mathrm{L}} = \sqrt{-\mathrm{d}\mathcal{T}/\mathrm{d}\mathcal{U}}$,
where $\mathcal{T} = -L$ and $\mathcal{U} = -\Lambda$ when $w < 0$,
while $\mathcal{T} = -\Lambda$ and $\mathcal{U} = -L$ when $w > 0$.

In all cases phase gradient proportional to surface current, $\overline{c}^{\mu} = x^{\mu}_{\ ,i}\overline{c}^{i}$,
$\overline{c}^{i} = \kappa\gamma^{ij}\varphi_{,j} = -\partial L \partial\varphi_{,i}$, that is conserved, $(\sqrt{-\gamma}\,\overline{c}^{i})_{,i} = 0$,
when no ext force, so that $\overline{T}^{\mu\nu} K_{\mu\nu}{}^{\rho} = 0$ with
$\overline{T}^{\mu\nu} = 2\kappa^{-1}\overline{c}^{\mu}\overline{c}^{\nu} + L\eta^{\mu\nu}$.

4. Free motion of circular elastic string loops

Circular loop invariant under action of axisymmetry generating Killing vector $\varrho^\mu\partial/\partial x^\mu = \partial/\partial\phi$ has conserved phase winding number $N = \varrho^\nu\overline{\nabla}_\nu\varphi$ and charge number $C = 2\pi\varrho^\mu\mathcal{E}_{\mu\nu}\bar{c}^\nu$, with $\mathcal{E}^{\mu\nu} = 2u^{[\mu}\tilde{u}^{\nu]}$, giving relation $w\varrho^2 = N^2 - (C/2\pi\kappa)^2$ between radius $\varrho = \sqrt{\varrho^\nu\varrho_\nu}$ and w, whose sign is determined by ratio $b = 2\pi\kappa_0 N/C$. Product is angular momentum $J = NC = \varrho^\nu\mathit{\Pi}_\nu$, where $\mathit{\Pi}^\mu = 2\pi\varrho^\nu\mathcal{E}_{\nu\rho}\overline{T}^{\rho\mu}$. Will also have conserved mass $M = -k^\nu\mathit{\Pi}_\nu$ in stationary background with Killing vector $k^\mu\partial/\partial x^\mu = \partial/\partial t$. Proper time variation of $\dot{\varrho}$ then given (in flat background) by $M^2\dot{\varrho}^2 = M^2 - \Upsilon^2$ with field Υ given implicitly via w as function of ϱ by $\Upsilon = C^2/2\pi\kappa\varrho - 2\pi L\varrho$.

5. Logarithmic equation of state for cosmic string

Linear action formula, $L = -m^2(1+\delta_*^2\, w)$ proposed for weak current by E.Witten, but no good – since would give subsonic $v^2 < v_{\rm L}^2$, contrary to effect of Witten's conduction mechanism as calculated by P. Peter. More realistic supersonic wiggle propagation $v^2 > v_{\rm L}^2$ from logarithmic formula

$L = -m^2 - \frac{1}{2} m_\star^2 \ln\{1+\delta_\star^2\, w\} \;\Rightarrow\; \kappa = m_\star^2 \delta_\star^2/(1+\delta_\star^2 w)\,,$

valid over finite range, $\exp\{-2m^2/m_*^2\} < 1+\delta_\star^2\, w < 2$. Radial dependence in circular case given explicitly by

$1/\kappa = 2(\pi m_*/rC)^2(-\varrho^2 + \sqrt{\varrho^4 + (C/\pi m_*^2)^2(\varrho^2/\delta_\star^2 + N^2)}\,.$

For ratio $|b|$ in "safe" range, $\exp\{-2m^2/m_*^2\} < |b| < 2$, (including "chiral" value $|b| = 1$) the ring can oscillate with unbounded energy M .

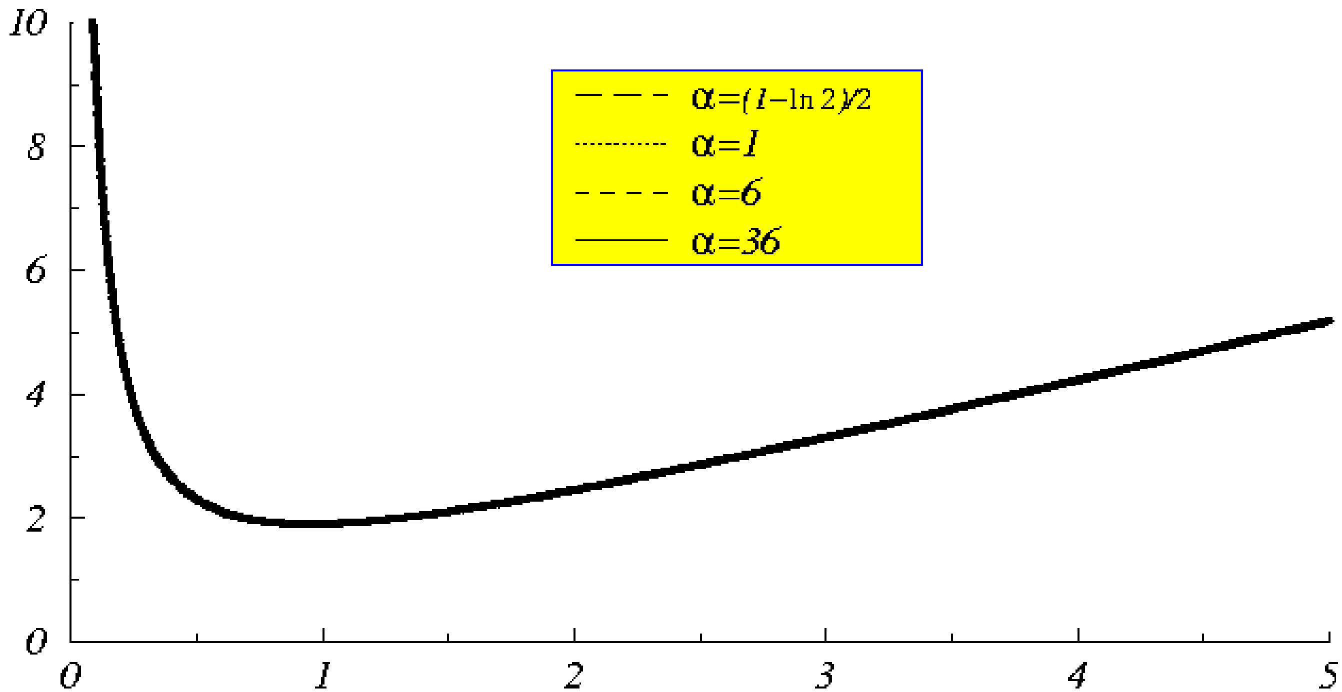

FIG. 1 – Plot of effective potential Υ against ϱ with various values of ratio $\alpha = m^2/m_\star^2$ for $|b| = 2\pi\kappa_0|N/C|$ in “safe” range not too far from chiral value $|b| = 1$. [hep-ph/9609401]

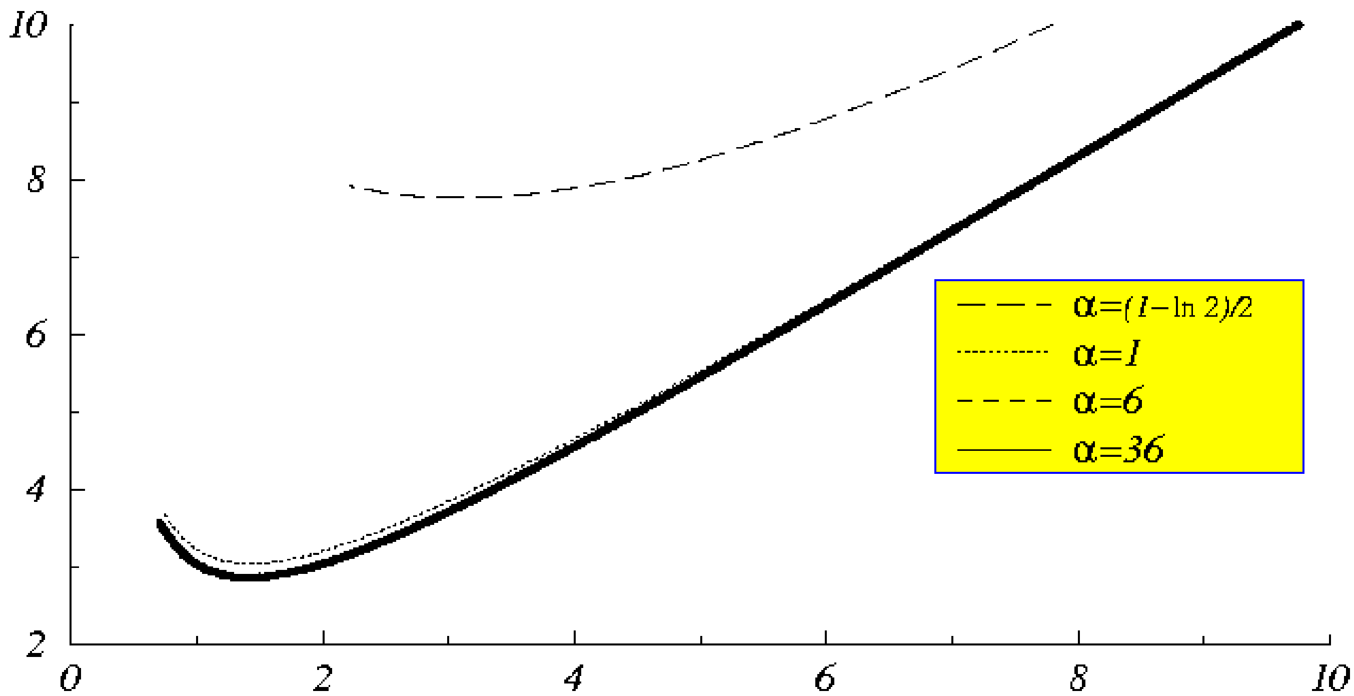

FIG. 2 – Plot of effective potential Υ against ϱ with various values of ratio $\alpha = m^2/m_\star^2$ for $|b| = 2\pi\kappa_0|N/C|$ outside "safe" range, where oscillation possible only for low values of energy M.

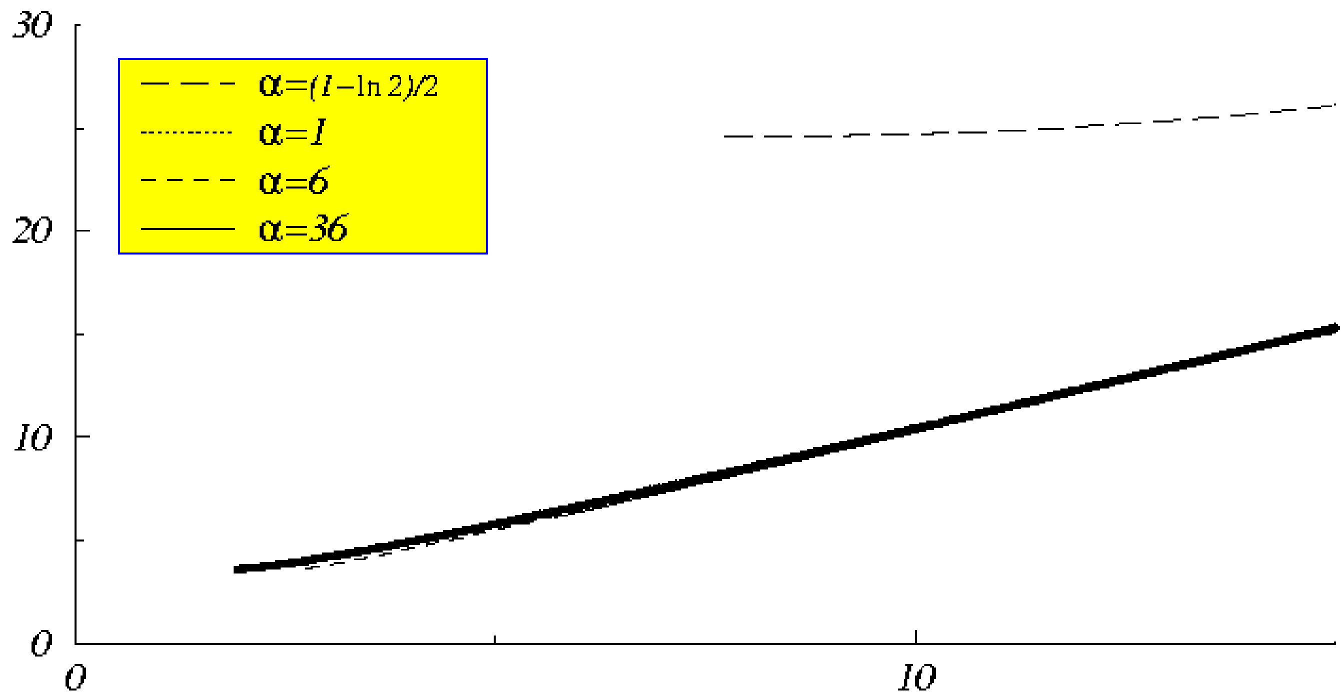

FIG. 3 – Plot of effective potential Υ against ϱ with various values of ratio $\alpha = m^2/m_\star^2$ for $|b| = 2\pi\kappa_0|N/C|$ far outside "safe" range, where no equilibrium nor oscillation is possible.

6. Stationary string states in flat background

World sheet tangent to unit static Killing vector satisfying $\nabla_\mu k^\nu = 0$, with orthog unit spacelike tangent vector e^μ satisfying $k^\nu \nabla_\nu e^\mu = 0$, giving first fundamental tensor $\eta^{\mu\nu} = -k^\mu k^\nu + e^\mu e^\nu$ and second fundamental tensor $K_{\mu\nu}{}^\rho = e_\mu e_\nu K^\rho$ with curvature vector $K^\mu = e^\nu \nabla_\nu e^\mu$. Flow velocity v of timelike eigenvector, $\overline{T}^\mu_\nu u^\nu = -\mathcal{U} u^\nu$, specified by expression $u^\mu = (1-v^2)^{-1/2}(k^\mu + v\, e^\mu)$.

Free dynamical equation reduces to $(\mathcal{U} - v^2\mathcal{T})K^\rho = 0$.

Straight solution, $K^\rho = 0$, possible for arbitrary v , but circular (or more general) curved configuration must have (generically uniform) wiggle propagation speed, $v^2 = \mathcal{T}/\mathcal{U}$.

7. Stability criterion for circular vorton states

Stability depends just on velocities v and v_{L}, always holds if subsonic $v^2 \leq v_{\mathrm{L}}^2$. Monopole $n = 0$ and dipole $n = 1$ modes always stable, but instability may occur for higher modes, $n \geq 2$ for which, in state with radius a, eigenfrequency ω given for $x = a\,\omega/v_{+}\,n$, with $v_{+} = 2v/(1+v^2)$, by cubic : $x^3 + b_2 x^2 + b_1 x + b_0 = 0$, in which $b_2 = \Gamma - 2 - \xi$, $b_1 = -2\Gamma + (1+\xi)\left(1 - n^{-2}\right)$, $b_0 = \Gamma\left(1 - n^{-2}\right)$, where $\xi = \Gamma(1 - v_{+}^2)$, $\Gamma = v_{+}^{-2}(v_{\mathrm{L}}^2 - v^2)/(1 - v_{\mathrm{L}}^2 v^2)$. Stability criterion $\Delta \geq 0$, for roots all real, given by $\Delta = b_2^2 b_1^2 + 18 b_2 b_1 b_0 - 4 b_1^3 - 4 b_2^3 b_0 - 27 b_0^2$. Relativistic limit $\xi \to 0$ gives $\Delta \to 4n^{-2}(\Gamma + 1 + n^{-1})^2(\Gamma + 1 - n^{-1})^2$, which is strictly positive almost always.

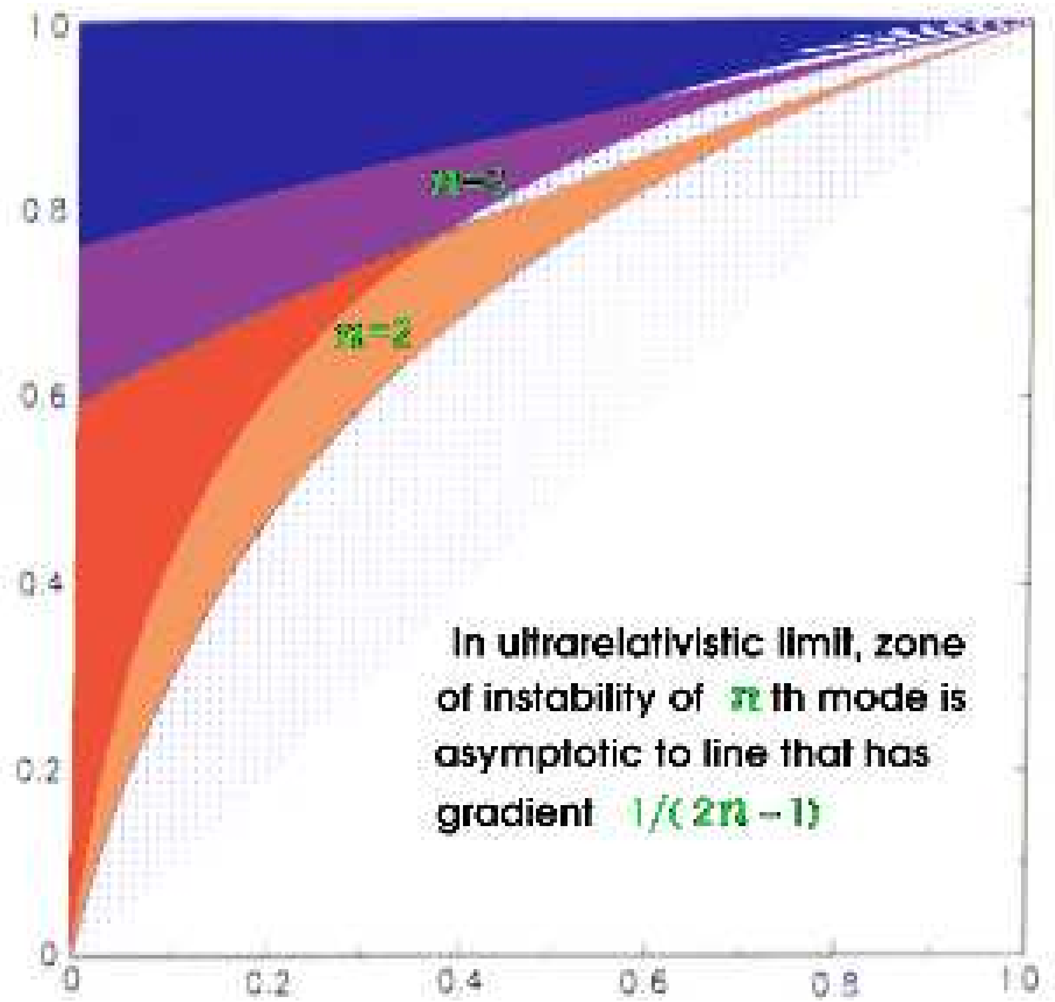

FIG. 4 – Zones of macroscopic instability of circular vorton states, as obtained in 1994 by X. Martin on plot of squared rotation (and wiggle) speed, v^2, against squared "sonic" speed $v_{\rm L}^2$.

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